Problem of the Month: *William's Polygons*

The Problems of the Month (POM) are used in a variety of ways to promote problem solving and to foster the first standard of mathematical practice from the Common Core State Standards: "Make sense of problems and persevere in solving them." The POM may be used by a teacher to promote problem solving and to address the differentiated needs of her students. A department or grade level may engage their students in a POM to showcase problem solving as a key aspect of doing mathematics. POMs can also be used school wide to promote a problem-solving theme at a school. The goal is for all students to have the experience of attacking and solving non-routine problems and developing their mathematical reasoning skills. Although obtaining and justifying solutions to the problems is the objective, the process of learning to problem solve is even more important.

The Problem of the Month is structured to provide reasonable tasks for all students in a school. The structure of a POM is a shallow floor and a high ceiling, so that all students can productively engage, struggle, and persevere. The Primary Version is designed to be accessible to all students and especially the key challenge for grades K – 1. Level A will be challenging for most second and third graders. Level B may be the limit of where fourth and fifth grade students have success and understanding. Level C may stretch sixth and seventh grade students. Level D may challenge most eighth and ninth grade students, and Level E should be challenging for most high school students. These grade- level expectations are just estimates and should not be used as an absolute minimum expectation or maximum limitation for students. Problem solving is a learned skill, and students may need many experiences to develop their reasoning skills, approaches, strategies, and the perseverance to be successful. The Problem of the Month builds on sequential levels of understanding. All students should experience Level A and then move through the tasks in order to go as deeply as they can into the problem. There will be those students who will not have access into even Level A. Educators should feel free to modify the task to allow access at some level.

Overview

In the Problem of the Month *William's Polygons,* students use properties of two-dimensional geometry to solve problems involving spatial visualization and generalization. The mathematical topics that underlie this POM are the attributes of polygons, symmetry, spatial visualization, transformations, designing patterns and shapes, using counting principles, similarity, ratios of geometric figures, and creating a general functional relationship to model the composition of geometric figures.

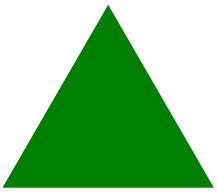
The problem asks the students to use spatial reasoning and attributes to make sense of geometric figures. In Level A of the POM, students are asked to decompose a

regular hexagon into equilateral triangles. The students use manipulatives or drawings to determine the number triangles. Then the students are asked to divide the hexagon into other common shapes that are all the same size. In level B, students are asked to explore the number of unit triangles that can cover a hexagon of different sizes. They determine the number of unit triangles needed to cover a hexagon with sides 2 units long and then another hexagon with sides 3 units in length. The students need to visualize and draw the figure in an effort to find all possible unit triangles. In level C, students are given a regular octagon and are asked to decompose the figure into congruent isosceles triangles whose vertex angle is at the center of the octagon. They are asked to find the measure of the vertex angles. Students then are challenged with finding the vertex and base angles of isosceles triangles in a decagon. Finally they are asked to generalize the measure of the vertex angle for any size polygon. In level D, the students explore how to create three quadrilaterals by drawing line segments from one vertex to two opposite vertices in an octagon. They need to determine shape and areas of the quadrilaterals. In level E, students are challenged to divide a triangle into only quadrilaterals. Students are asked to find a solution or prove that it cannot be done.

Mathematical Concepts

Spatial visualization plays an important part in real-world experiences. Whether designing the most complex structures created by designers, architects, and construction workers or arranging the furniture in a room, spatial awareness and visualization are essential. In this POM, students explore various aspects of spatial visualization. This involves examining symmetrical patterns as well as designing systematic ways to classify or categorize geometric arrangement of figures. Students will use their spatial sense and develop understandings of attributes in plane geometry. They will also design new geometric figures using basic shapes. In addition to the geometric aspects of this POM, students are seeking to find how many and how much, which requires skills in counting. The students will need to determine the perimeter, area, and ratio of the sizes of figures and determine how to compose and decompose shapes within a figure. The combination of spatial visualization, knowledge of geometric attributes, and counting principles will be called upon to solve the problems in this POM.

Problem of the Month
William's Polygons



Level A

William likes to play with the yellow regular hexagons. The hexagon has sides that are each one unit in length. He knows he can separate one hexagon into equilateral triangles that also have sides one unit in length.

How many equilateral triangles will be needed?

Show how.

Using other polygons, how else can the hexagon be separated?
Show the other possibilities and explain.

Level B

William has a regular hexagon with sides that are 2 units each. How many equilateral triangles with 1-unit sides can be put together to make the hexagon?

Show how you figured it out.

How many equilateral triangles with 1-unit sides can be put together to make a regular hexagon with sides that are each 3 units long?

Level C

Bill draws a regular octagon. He divides the octagon into eight congruent isosceles triangles. What are the measures of the three angles in the isosceles triangle?

Bill draws a regular decagon. He divides the decagon into ten congruent isosceles triangles. What are the measures of the three angles in the isosceles triangles?

Given any regular polygon with **n** sides that can be divided into congruent isosceles triangles, find a rule for determining the vertex angle of the isosceles triangles.

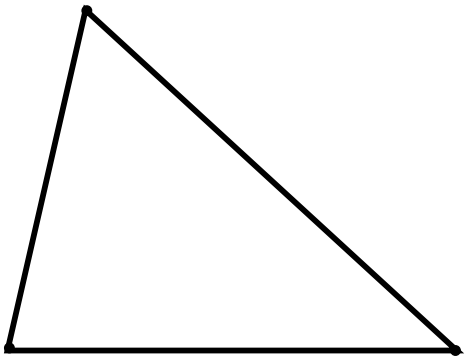
Level D

Bill draws a regular octagon with a perimeter of 32 units. He picks a vertex of the octagon and draws two line segments through the interior of the octagon, each to a different opposite vertex. His line segments divide the octagon into three quadrilaterals. What does the shape look like?

What are the areas of each of the three quadrilaterals?

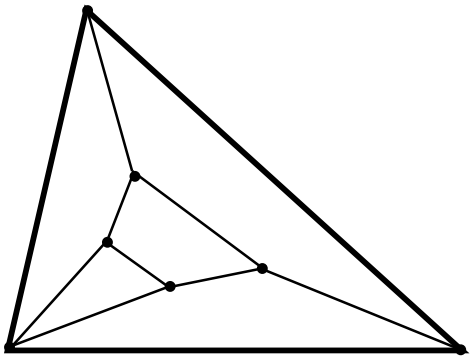
Level E

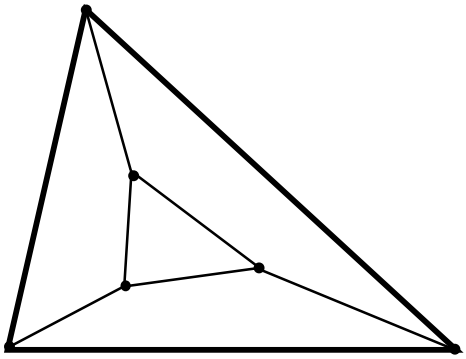
Mr. Medigovich drew the following triangle:



Mr. Medigovich asked his students, "If no new vertices are permitted on the edge of the original triangle, are you able to cut this triangle into quadrilaterals only?"

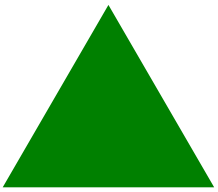
Two students made the following unsuccessful attempts:





He now asks you, "Can you cut this triangle into quadrilaterals only?" Explain your solution.

Problem of the Month
William's Polygons



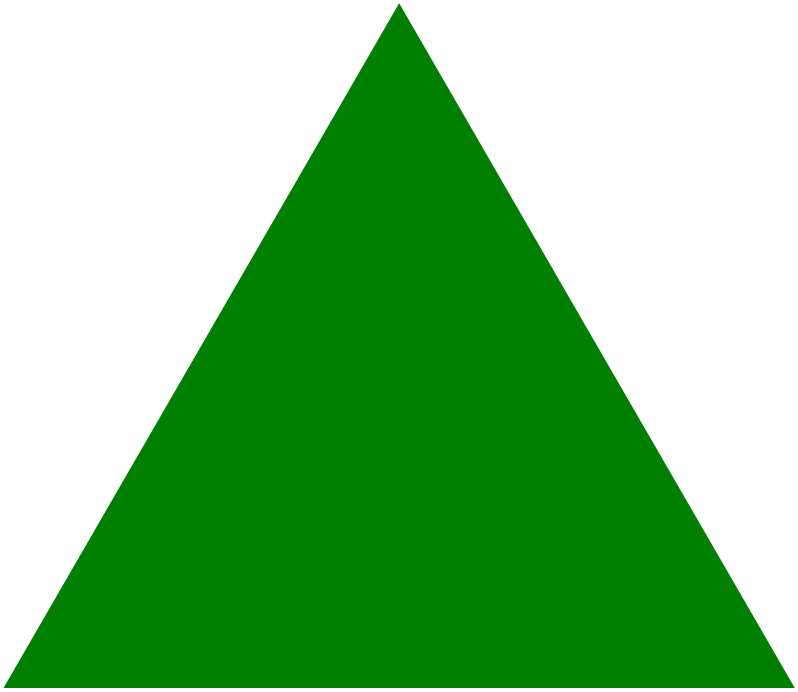
Primary Version Level A

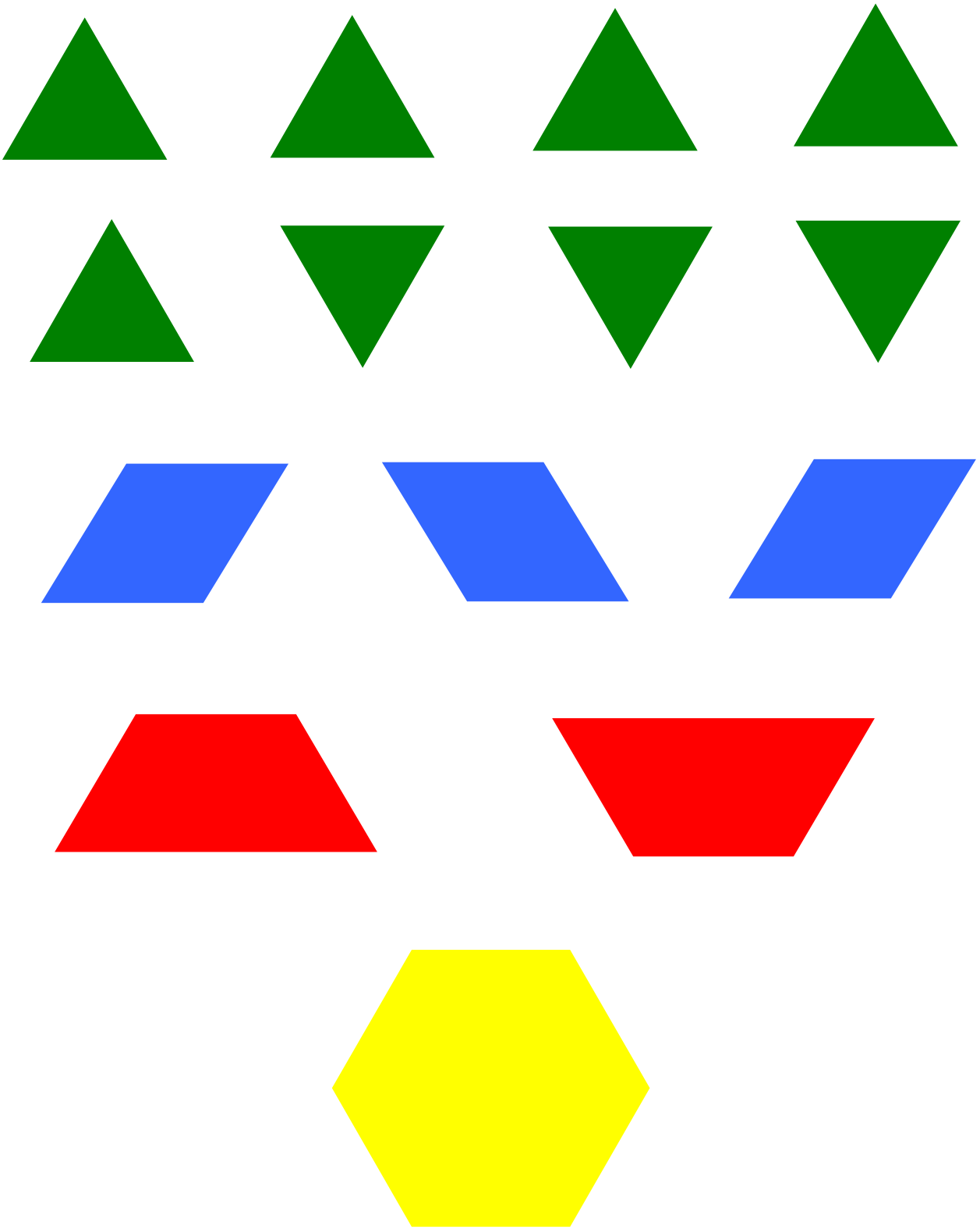
Materials. For the teacher: Picture of a blue rhombus pattern block and at least two green triangle pattern blocks. For each student: Red trapezoid, blue rhombi and green triangle pattern blocks, paper and pencil.

Discussion on the rug: Teacher shows the picture of a green triangle**.** **"What do we call this figure?"** Students volunteer what they see. Teacher asks, **"How many sides does it have?"** Students respond and show the sides. Teacher confirms by chorale counting. Teacher shows the picture of a blue rhombus. **"How many sides does this figure have?"** Teacher calls on students to explain. Teacher asks, **"Is there a way to use the triangles to make a blue rhombus?"** Class explores these ideas. After some individual think time followed by pair share, the teacher asks, **"Who can show me how?"** Teacher invites students to share and show their methods.

In small groups: Students work in pairs. Teacher passes out pattern blocks. The teacher says to the class, **"Who can make the red trapezoid shape out of other pieces? Show how many ways you can do it?"**

At the end of the investigation have students either discuss or dictate a response to this summary question: **"Tell me how you can make the same trapezoid shape using other shapes. Tell me how you figured it out."**





Problem of the Month

William's Polygons

Task Description – Level A

This task asks students to use spatial reasoning and geometric attributes to make sense of a figure. The students are asked to decompose a regular hexagon into equilateral triangles. The students use manipulatives or drawings to determine the number of triangles. Then the students are asked to divide the hexagon into other common shapes that are all the same size. .

Common Core State Standards Math - Content Standards

Geometry

Reason with shapes and their attributes.

1.G.1. Distinguish between defining attributes versus non-defining attributes; build and draw shapes to possess defining attributes.

1.G.2. Compose two-dimensional shapes (rectangles, squares, trapezoids, triangles, half-circles, and quarter-circles) or three-dimensional shapes (cubes, right rectangular prisms, right circular cones, and right circular cylinders) to create a composite shape, and compose new shapes from the composite shape.

2.G.1. Recognize and draw shapes having specified attributes, such as a given number of angles or a given number of equal faces.

Common Core State Standards Math – Standards of Mathematical Practice

MP.1 Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

MP.7 Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7 x 8 equals the well-remembered 7 x 5 + 7 x 3, in preparation for learning about the distributive property. In the expression $\mathbf{x}^2 + 9\mathbf{x} + 14$, older students can see the 14 as 2 x 7 and the 9 as 2 + 7. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(\mathbf{x} - \mathbf{y})^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers **x** and **y**.

Problem of the Month

William's Polygons

Task Description – Level B

This task challenges a student to explore the number of unit triangles that can cover a hexagon of different sizes. They determine the number of unit triangles needed to cover a hexagon with sides 2 units long and then another hexagon with sides 3 units in length. The students need to visualize and draw the figure in an effort to find all possible unit triangles.

Common Core State Standards Math - Content Standards

Geometry

Reason with shapes and their attributes.

1.G.1. Distinguish between defining attributes versus non-defining attributes; build and draw shapes to possess defining attributes.

1.G.2. Compose two-dimensional shapes or three-dimensional shapes to create a composite shape, and compose new shapes from the composite shape.

2.G.1.Recognize and draw shapes having specified attributes, such as a given number of angles or a given number of equal faces.

3.G.2. Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole.

Common Core State Standards Math – Standards of Mathematical Practice

MP.4 Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose..

MP.7 Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7 x 8 equals the well-remembered 7 x 5 + 7 x 3, in preparation for learning about the distributive property. In the expression $\mathbf{x}^2 + 9\mathbf{x} + 14$, older students can see the 14 as 2 x 7 and the 9 as 2 + 7. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 – 3(\mathbf{x} – \mathbf{y})^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers **x** and **y**.

Problem of the Month
William's Polygons
Task Description – Level C
This task challenges a student to explore the central angles in regular polygons. The students are given a regular octagon and are asked to decompose the figure into congruent isosceles triangles whose vertex angle is at the center of the octagon. They are asked to find the measure of the vertex angles. Students then are challenged with finding the vertex and base angles of isosceles triangles in a decagon. Finally they are asked to generalize the measure of the vertex angle for any size polygon.
Common Core State Standards Math - Content Standards
Geometry **Reason with shapes and their attributes.** 3.G.2. Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole. **Draw and identify lines and angles, and classify shapes by properties of their lines and angles.** 4.G.1. Draw points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines. Identify these in two-dimensional figures. **Operations and Algebraic Thinking** **Generate and analyze patterns.** 4.OA.5. Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself.
Common Core State Standards Math – Standards of Mathematical Practice
MP.7 Look for and make use of structure. Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7 x 8 equals the well-remembered 7 x 5 + 7 x 3, in preparation for learning about the distributive property. In the expression $\mathbf{x}^2 + 9\mathbf{x} + 14$, older students can see the 14 as 2 x 7 and the 9 as 2 + 7. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 – 3(\mathbf{x} – \mathbf{y})^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers **x** and **y**. **MP.8 Look for and express regularity in repeated reasoning.** Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y – 2)/(x – 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x – 1)(x + 1)$, $(x – 1)(x2 + x + 1)$, and $(x – 1)(x3 + x2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Problem of the Month:

William's Polygons

Task Description – Level D

This task challenges a student to use visualization and reasoning to construct and draw quadrilaterals with different attributes. The students explore how to create three quadrilaterals by drawing line segments from one vertex to two opposite vertices in an octagon. They need to determine shape and areas of the quadrilaterals.

Common Core State Standards Math - Content Standards

Geometry

Draw and identify lines and angles, and classify shapes by properties of their lines and angles.
4.G.1.Draw points, lines, line segments, rays, angles (right, acute, obtuse), and perpendicular and parallel lines. Identify these in two-dimensional figures.

Draw, construct, and describe geometrical figures and describe the relationships between them.
7.G.2. Draw (freehand, with ruler and protractor, and with technology) geometric shapes with given conditions.

Solve real-life and mathematical problems involving angle measure, area, surface area, and volume.
7.G.6. Solve real-world and mathematical problems involving area, volume and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.

Common Core State Standards Math – Standards of Mathematical Practice

MP.1 Make sense of problems and persevere in solving them.
Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

MP.3 Construct viable arguments and critique the reasoning of others.
Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

Problem of the Month
William's Polygons
Task Description – Level E
This task challenges a student to analyze a pattern involving geometric shapes. The students are challenged to divide a triangle into only quadrilaterals. Students are asked to find a solution or prove that it cannot be done..
Common Core State Standards Math - Content Standards
High School – Geometry - Modeling with Geometry **Apply geometric concepts in modeling situations** G-MG. 1. Use geometric shapes, their measures, and their properties to describe objects. 3. Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).
Common Core State Standards Math – Standards of Mathematical Practice
MP.4 Model with mathematics. Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose. **MP.7 Look for and make use of structure.** Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7 x 8 equals the well-remembered 7 x 5 + 7 x 3, in preparation for learning about the distributive property. In the expression $\mathbf{x}^2 + 9\mathbf{x} + 14$, older students can see the 14 as 2 x 7 and the 9 as 2 + 7. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 – 3(\mathbf{x} – \mathbf{y})^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers **x** and **y**. **MP.8 Look for and express regularity in repeated reasoning.** Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y – 2)/(x – 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x – 1)(x + 1)$, $(x – 1)(x2 + x + 1)$, and $(x – 1)(x3 + x2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Problem of the Month

William's Polygons

Task Description – Primary Level

This task asks students to use spatial reasoning and geometric attributes to make sense of a figure. The students are asked to decompose a rhombus into equilateral triangles. The students use manipulatives or drawings to determine the number triangles. Then the students are asked to divide the trapezoid into triangles and triangles and a rhombus.

Common Core State Standards Math - Content Standards

Geometry

Identify and describe shapes

K.G.2. Correctly name shapes regardless of their orientations or overall size.

Analyze, compare, create, and compose shapes.

K.G.4. Analyze and compare two- and three-dimensional shapes, in different sizes and orientations, using informal language to describe their similarities, differences, parts.

K.G.5. Model shapes in the world by building shapes from components and drawing shapes.

K.G.6. Compose simple shapes to form larger shapes.

Common Core State Standards Math – Standards of Mathematical Practice

MP.1 Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

MP.2 Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

MP.7 Look for and make use of structure.

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