Problem of the Month: *Infinite Windows*

The Problems of the Month (POM) are used in a variety of ways to promote problem solving and to foster the first standard of mathematical practice from the Common Core State Standards: "Make sense of problems and persevere in solving them." The POM may be used by a teacher to promote problem solving and to address the differentiated needs of her students. A department or grade level may engage their students in a POM to showcase problem solving as a key aspect of doing mathematics. POMs can also be used school wide to promote a problem-solving theme at a school. The goal is for all students to have the experience of attacking and solving non-routine problems and developing their mathematical reasoning skills. Although obtaining and justifying solutions to the problems is the objective, the process of learning to problem solve is even more important.

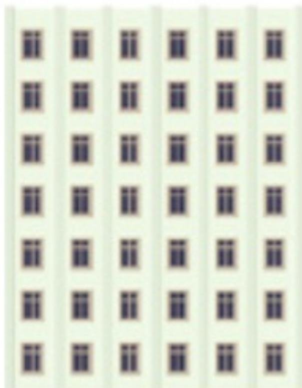
The Problem of the Month is structured to provide reasonable tasks for all students in a school. The structure of a POM is a shallow floor and a high ceiling, so that all students can productively engage, struggle, and persevere. The Primary Version is designed to be accessible to all students and especially the key challenge for grades K – 1. Level A will be challenging for most second and third graders. Level B may be the limit of where fourth and fifth grade students have success and understanding. Level C may stretch sixth and seventh grade students. Level D may challenge most eighth and ninth grade students, and Level E should be challenging for most high school students. These grade- level expectations are just estimates and should not be used as an absolute minimum expectation or maximum limitation for students. Problem solving is a learned skill, and students may need many experiences to develop their reasoning skills, approaches, strategies, and the perseverance to be successful. The Problem of the Month builds on sequential levels of understanding. All students should experience Level A and then move through the tasks in order to go as deeply as they can into the problem. There will be those students who will not have access into even Level A. Educators should feel free to modify the task to allow access at some level.

Overview

In the Problem of the Month, *Infinite Windows,* students use properties of two-dimensional geometry to solve problems involving spatial visualization. The mathematical topics that underlie this POM are the attributes of polygons, symmetry, spatial visualization, transformations, patterns, functions and fractal geometry.

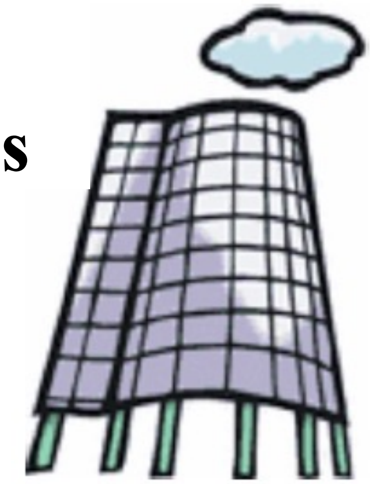
The problem asks the student to use spatial reasoning to make sense of part of a visual image. In the first level of the POM, students are asked to determine the number of rectangles in a figure comprised of different-sized rectangles. Their task is to identify the number of rectangles and show how they were counted. In level B, students are presented with a similar situation that involves finding the number of triangles in a figure comprised of different-sized triangles. The students are also asked to compare the length and area of the different-sized triangles. In level C, students search for ways to tile figures with self-similar shapes upon the original figure. The students must determine which of the figuress can be tiled and which figures cannot. They are asked to explain their findings. In level D, the students explore a visual infinite sequence created looking through opposite reflecting mirrors. The image of a woman is repeated, decreasing in size. The students determine the change in the size of the women at various stages of the visual image. In level E, the students explore a geometric fractal image. The fractal image is generated from creating square holes of differing sizes in a square. The process is repeated and the students draw the image, find the area of the new figure, and the perimeter of all edges at each stage. Using knowledge of patterns and functions, the students predict what the fractal looks like and its size in terms of area and perimeter at any stage of the fractal's development.

Mathematical Concepts
Spatial visualization plays an important part in real-world experiences. From designing the most complex structures created by designers, architects, and construction workers to arranging the furniture in a room, spatial awareness and visualization are essential. In this POM, students explore various aspects of spatial visualization. This involves examining symmetrical patterns as well and understanding the relationship between multiple lines of reflection. Students will use their spatial sense and develop understandings of attributes in plane geometry and the use of basic transformations. In addition to the geometric aspects of this POM, the students are seeking to find patterns, develop functional relationships and examine the measurement aspect of perimeter and area in fractals. The mathematics involved in the higher level is fractal geometry and chaos theory.

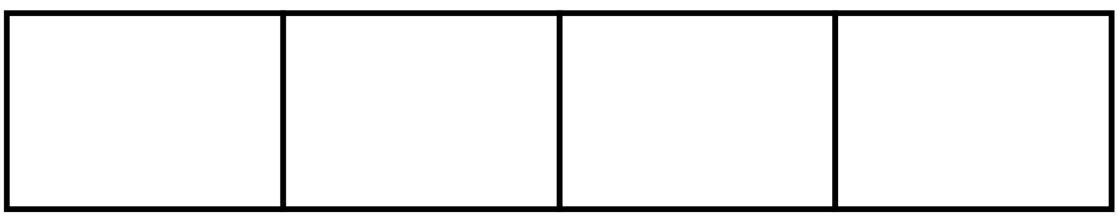


Problem of the Month

Infinite Windows



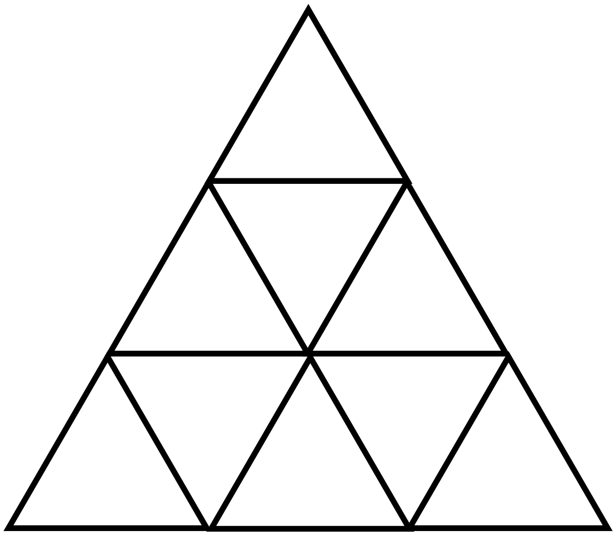
Level A
Tracy and Debbie are counting rectangles. The figure below is made of different rectangles and rectangles of different sizes.



How many rectangles can you find in the figure above? Show and explain how you found all of them.

Level B

Ramis and Aaron are counting triangles. The figure below is made of several equilateral triangles of different sizes.



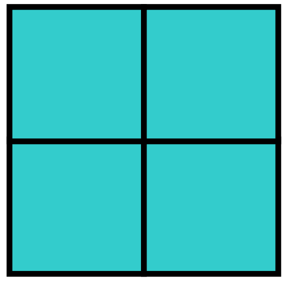
How many triangles can be found in the figure above? Explain your findings.

How do the sizes of the triangles compare? How do they compare in terms of area? How do they compare in terms of length?

If one more row were added to the bottom of the figure, how many triangles would be found in the new figure?

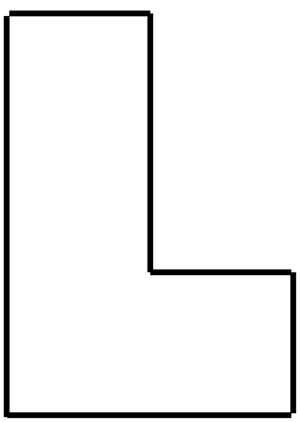
Level C

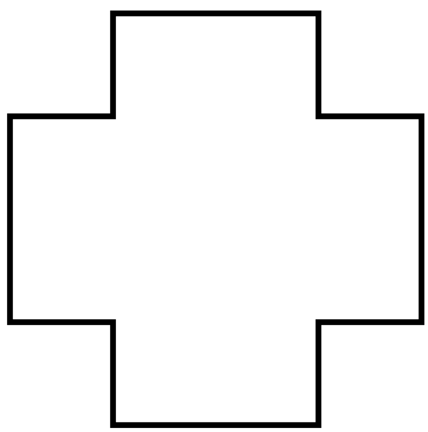
A *Perfect Polygon Puzzle* is a set of congruent polygon pieces that form to make a larger polygon similar to the puzzle pieces. For example, a square can be a perfect polygon puzzle since four squares can be arranged to make a larger square, as shown below:

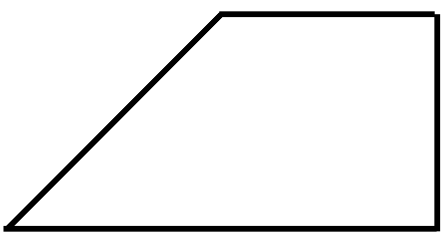


Do all regular polygons have this property? Explain your answer fully.

Other non-regular polygons can be *perfect polygon puzzles*. Which of the figures below are *perfect polygon puzzles*. If they are, show how they are. If they are not, explain why.







Level D
Have you ever sat in a hair salon or barbershop and looked through opposite mirrors? You can look down an endless tunnel of images that are all the same. What you see is the front of your head and the back of your head in a series of identical pictures, appearing smaller and smaller, forever. Likewise, if you look at a picture of a girl holding a mirror while looking in a mirror, you experience the same thing. Examine the picture of the girl with the mirror below:

How do the heights of the girl in the smaller pictures compare to the height of the girl in the original picture?

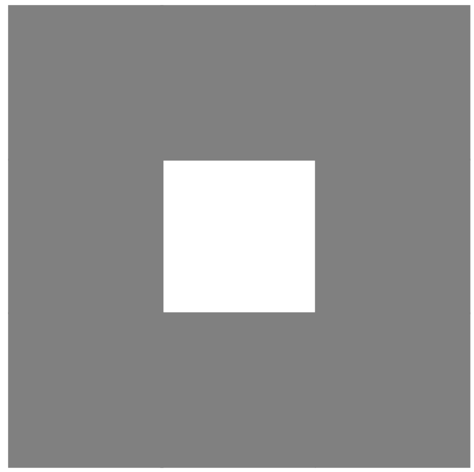
If you had a microscope and could zoom in on the tenth picture of the girl, what would be the actual height of the girl in that picture? Explain.

Suppose you could use a microscope to see something 0.0001 millimeter tall. How many pictures of the girl would you be able to see then? Explain.

Level E

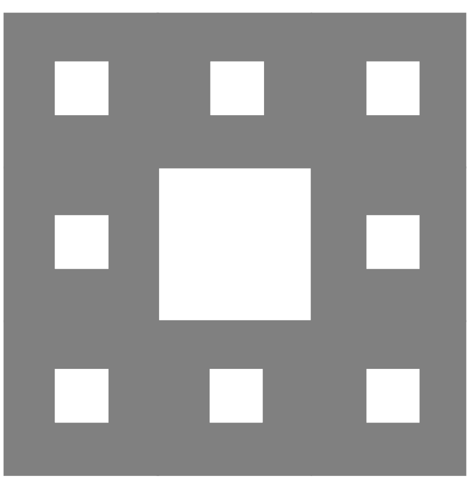
Sierpinski's Carpet is a fractal. It is created in stages. The initial stage (Stage 0) is a square. Stage 1 has a square hole cut out of the center of the original square exactly one ninth of the area of the original square.

Stage 0



Stage 1

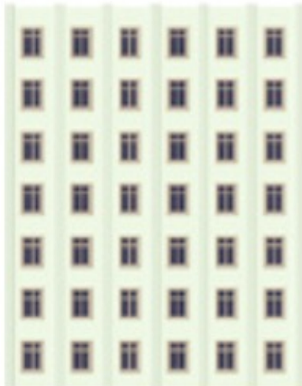
Stage 2 is created from Stage 1. It has eight new smaller square holes cut out of the remaining area. Where do those holes appear to be on the original square and how big are the square holes? State the area of the small hole in terms of the fractional part of the original square (Stage 0). If the original square was 1 square unit in area, what fractional part of the area of the square remains in Stage 2? Explain.



Stage 2

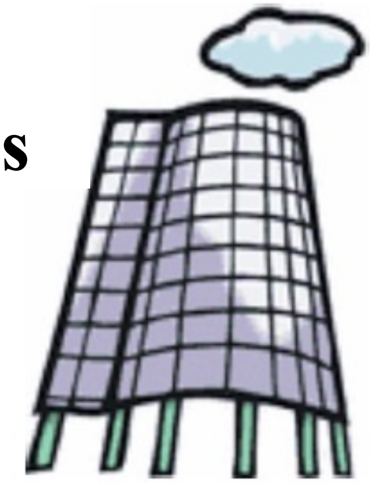
- What do the next three stages of the fractal look like (draw or describe)?
- At Stage 5, what area of the original square remains?
- If the perimeter of the fractal includes all the lengths of borders of the square, including the outside, as well as the length that surrounds each hole, what would be the perimeter of each stage (0 to 5) considering the length of the side of Stage 0 as one unit in length?

- Since all fractals have an infinite number of stages, what do you think the area approaches and the perimeter approaches as you get larger and larger stage numbers? Explain.



Problem of the Month

Infinite Windows

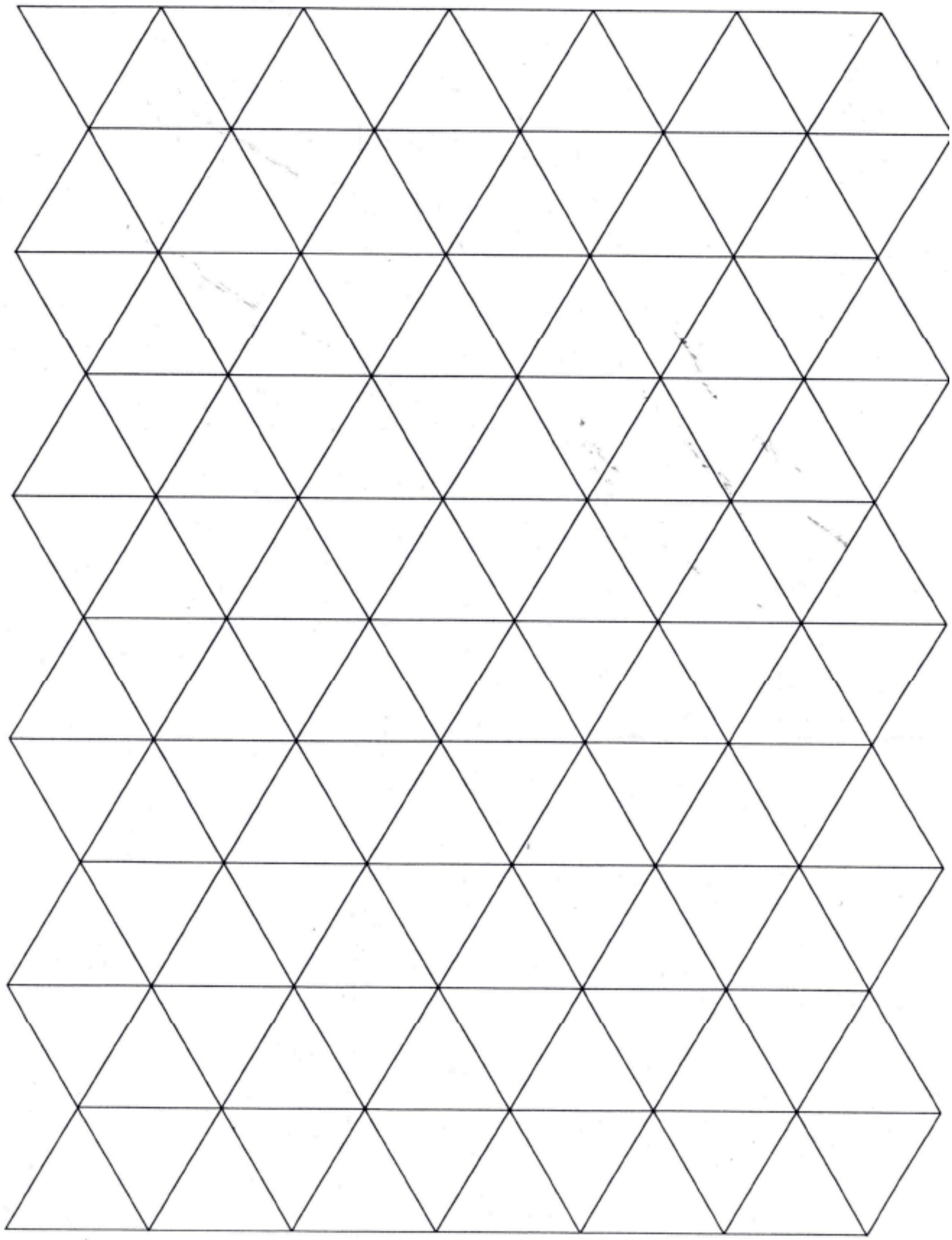


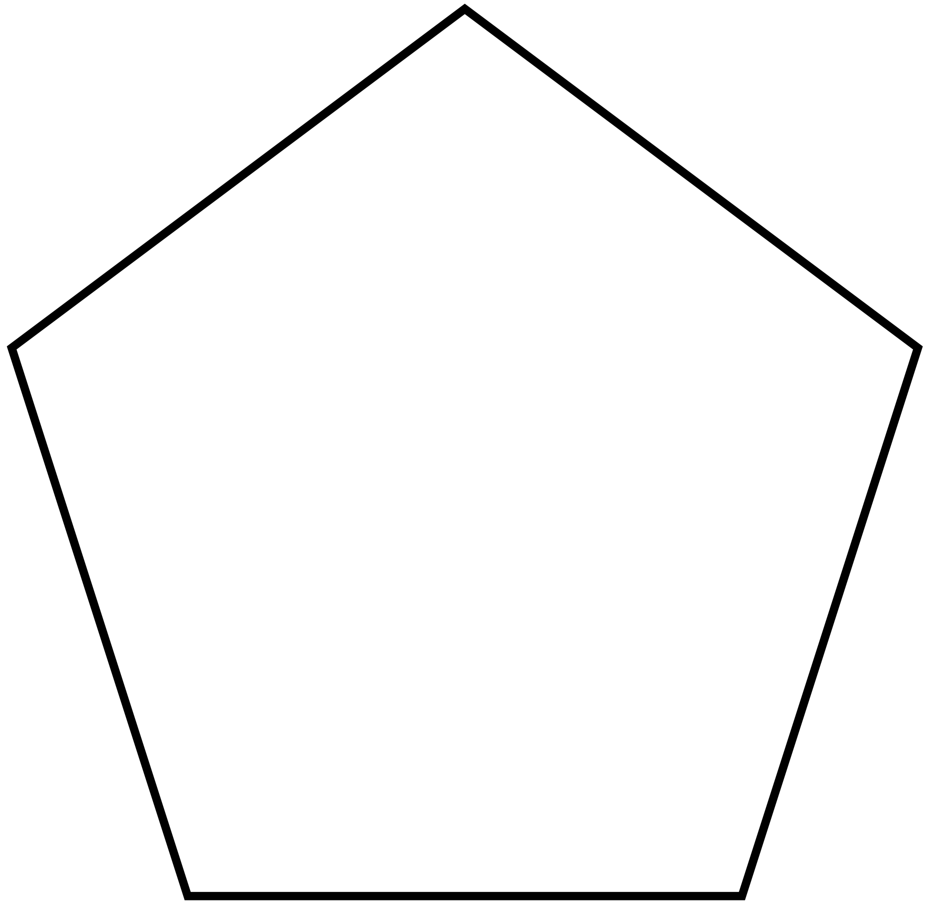
Primary Version Level A

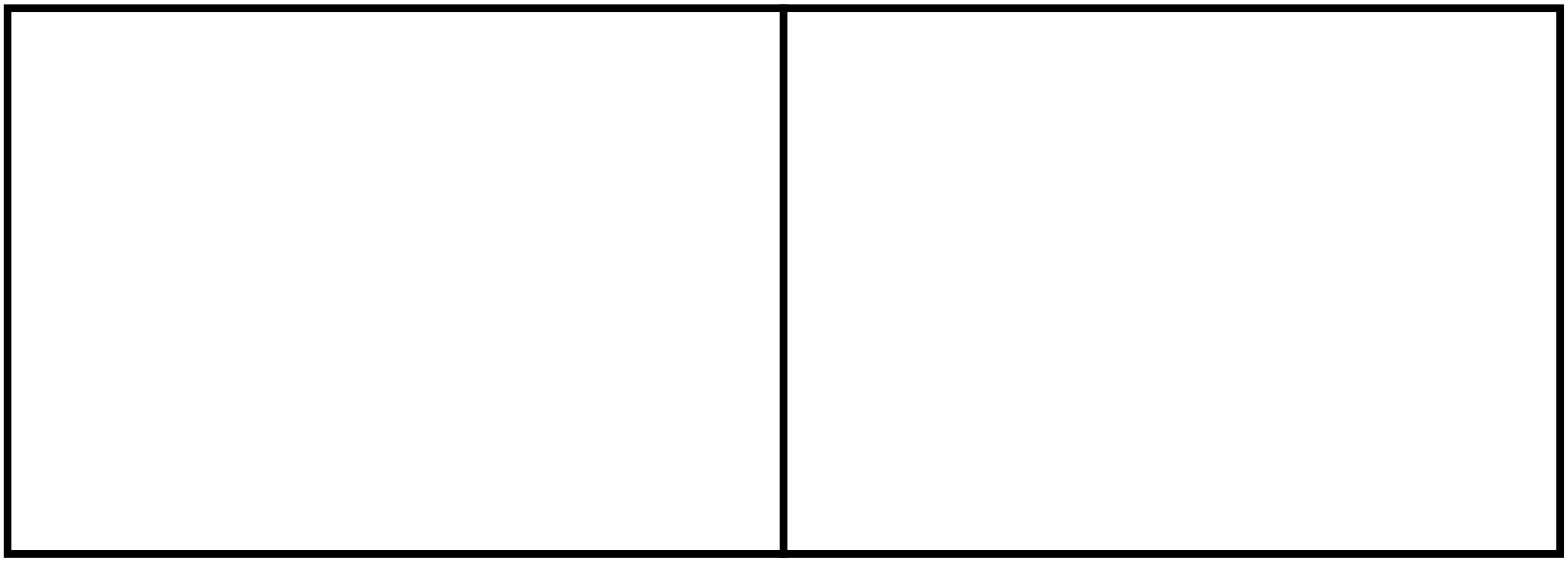
Materials: For demonstrations, pictures of a rectangle, pentagon, parallelogram, and two adjacent rectangles. For students, three adjacent rectangles, paper and pencil.

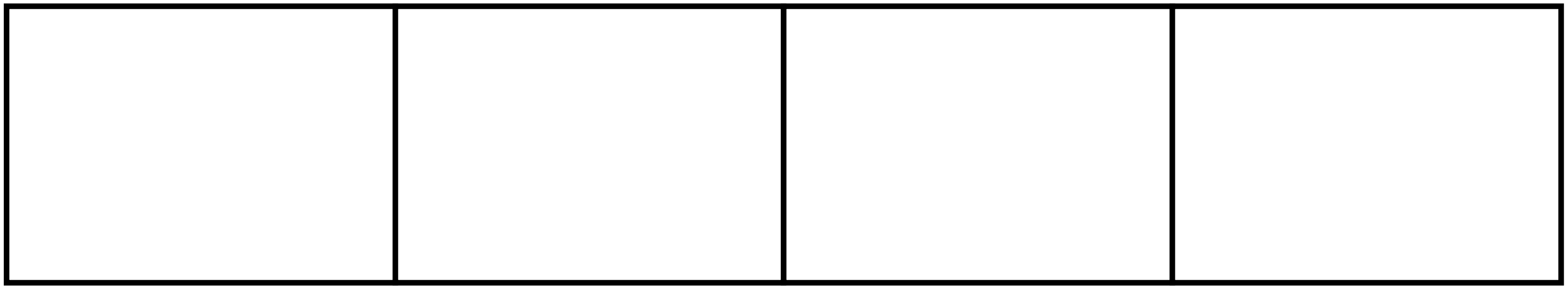
Discussion on the rug: Teacher asks the class. **"Who knows what a rectangle looks like?"** Students respond. The teacher shows the class a rectangle and asks students to explain the parts of the rectangle. The teacher shows a picture of a pentagon and asks if this figure is a rectangle. Students respond. The teacher clarifies why a pentagon is not a rectangle. The teacher repeats the question and explanation as she holds up a parallelogram. Teacher states, **"I am going to show you a picture. I want you to look for rectangles in the picture."** Teacher holds the picture of a pair of small rectangles. **"How many rectangles can you find?"** Students respond and teacher probes to help students see three.

In small groups: Each student has access to the sheet with three adjacent rectangles, pencil and paper. The teacher asks them to find all the rectangles in the picture. Teacher encourages them to draw or label to keep track.
At the end of the investigation, the teacher asks students either to discuss or to dictate a response to the prompt:
"How many rectangles do you see in the picture? How do you know you found them all?"









Problem of the Month
Infinite Windows
Task Description – Level A
This task asks students to use spatial reasoning and geometric attributes to make sense of a figure. The students are asked to determine the number of rectangles in a figure comprised of different-size rectangles. Their task is to identify the number of rectangles and show how they were counted.
Common Core State Standards Math - Content Standards
<u>Geometry</u> **Reason with shapes and their attributes.** 1.G.1. Distinguish between defining attributes versus non-defining attributes; build and draw shapes to possess defining attributes. 1.G.2. Compose two-dimensional shapes (rectangles, squares, trapezoids, triangles, half-circles, and quarter-circles) or three-dimensional shapes (cubes, right rectangular prisms, right circular cones, and right circular cylinders) to create a composite shape, and compose new shapes from the composite shape. 2.G.1. Recognize and draw shapes having specified attributes, such as a given number of angles or a given number of equal faces. **<u>Geometry</u>** **Reason with shapes and their attributes.** 2.G.2 Partition a rectangle into rows and columns of the same-size squares and count to find the total number of them.
Common Core State Standards Math – Standards of Mathematical Practice
MP.1 Make sense of problems and persevere in solving them. Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches. **MP.7 Look for and make use of structure.** Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7 x 8 equals the well-remembered 7 x 5 + 7 x 3, in preparation for learning about the distributive property. In the expression $\mathbf{x}^2 + 9\mathbf{x} + 14$, older students can see the 14 as 2 x 7 and the 9 as 2 + 7. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(\mathbf{x} - \mathbf{y})^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers **x** and **y**.

Problem of the Month
Infinite Windows
Task Description – Level B
This task challenges a student to explore the number of unit triangles that can cover a larger triangle. The students are presented with a problem that involves finding the number of triangles in a figure comprised of different-size triangles. The students are also asked to compare the length and area of the different-size triangles.
Common Core State Standards Math - Content Standards
<u>Geometry</u> **Reason with shapes and their attributes.** 3.G.2. Partition shapes into parts with equal areas. Express the area of each part as a unit fraction of the whole. **<u>Measurement and Data</u>** **Geometric measurement: understand concepts of area and relate area to multiplication and to addition.** 3.MD.5 Recognize area as an attribute of plane figures and understand concepts of area measurement. a. A square with side length 1 unit, called "a unit square," is said to have "one square unit" of area, and can be used to measure area. b. A plane figure which can be covered without gaps or overlaps by n unit squares is said to have an area of n square units. **<u>Operations and Algebraic Thinking</u>** **Generate and analyze patterns.** 4.OA.5. Generate a number or shape pattern that follows a given rule. Identify apparent features of the pattern that were not explicit in the rule itself.
Common Core State Standards Math – Standards of Mathematical Practice
MP.4 Model with mathematics. Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.. **MP.7 Look for and make use of structure.** Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7 x 8 equals the well-remembered 7 x 5 + 7 x 3, in preparation for learning about the distributive property. In the expression $\mathbf{x}^2 + 9\mathbf{x} + 14$, older students can see the 14 as 2 x 7 and the 9 as 2 + 7. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(\mathbf{x} - \mathbf{y})^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers **x** and **y**.

Problem of the Month
Infinite Windows
Task Description – Level C
This task challenges a student to explore self-similarity. The students search for ways to tile figures with self-similar shapes upon the original figure. The students must determine which of the files can be tiled and which figures cannot. They are asked to explain their findings.
Common Core State Standards Math - Content Standards
Geometry **Understand congruence and similarity using physical models, transparencies, or geometry software.** 8.G.4 Understand that a two-dimensional figure is similar to another if the second can be obtained from the fist by a sequence of rotations, reflections, translations, and dilations; given two similar two-dimensional figures, describe a sequence that exhibits the similarity between them. **Ratios and Proportional Relationships** **Understand ratio concepts and use ratio reasoning to solve problems.** 6.RP.3 Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.
Common Core State Standards Math – Standards of Mathematical Practice
MP.7 Look for and make use of structure. Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7 x 8 equals the well-remembered 7 x 5 + 7 x 3, in preparation for learning about the distributive property. In the expression $\mathbf{x}^2 + 9\mathbf{x} + 14$, older students can see the 14 as 2 x 7 and the 9 as 2 + 7. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 – 3(\mathbf{x} – \mathbf{y})^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers **x** and **y**. **MP.8 Look for and express regularity in repeated reasoning.** Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation (y – 2)/(x – 1) = 3. Noticing the regularity in the way terms cancel when expanding (x – 1)(x + 1), (x – 1)(x2 + x + 1), and (x – 1)(x3 + x2 + x + 1) might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Problem of the Month:
Infinite Windows
Task Description – Level D
This task challenges a student to use visualization and reasoning to model self-similar relationships. The students explore a visual infinite sequence created looking through opposite reflecting mirrors. The image of a woman is repeated, decreasing in size. The students determine the change in the size of the women at various stages of the visual image.
Common Core State Standards Math - Content Standards
<u>**Geometry**</u> **Understand congruence and similarity using physical models, transparencies, or geometry software.** 8.G.4 Understand that a two-dimensional figure is similar to another if the second can be obtained from the fist by a sequence of rotations, reflections, translations, and dilations; given two similar two-dimensional figures, describe a sequence that exhibits the similarity between them. <u>**Expressions and Equations**</u> **Solve real-life and mathematical problems using numerical and algebraic expressions equations.** 7.EE.3 Solve multi-step real-life and mathematical problems posed with positive and negative rational numbers in any form (whole numbers, fractions, and decimals), using tools strategically. Apply properties of operations to calculate with numbers in any form; convert between forms as appropriate; and assess the reasonableness of answers using mental computation and estimation strategies. 7.EE.4 Use variable to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems by reasoning about the quantities. **Use functions to model relationships between quantities.** 8.EE.4 Construct a function to model a linear relationship between two quantities Determine the rate of change and the initial value of the function from a description of a relationship or from two (x,y) values, including reading these from a table or from a graph; interpret the rate of change and initial value of a linear function in terms of the situation it models, and in forms of the graph or table of values.
Common Core State Standards Math – Standards of Mathematical Practice
MP.1 Make sense of problems and persevere in solving them. Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

Problem of the Month

Infinite Windows

Task Description – Level E

The students explore a geometric fractal image. The fractal image is generated from creating square holes of differing sizes in a square. The process is repeated and the students draw the image, find the area of the new figure, and the perimeter of all edges at each stage. Using knowledge of patterns and functions, the students predict what the fractal looks like and its size in terms of area and perimeter at any stage of the fractal's development.

Common Core State Standards Math - Content Standards

High School – Algebra – Creating Equations

Create equations that describe members or relationships.

A-CED.1 Create equations and inequalities in one variable and use them to solve problems, include equations arising from linear and quadratic functions, and simple rational and exponential functions.

High School – Geometry – Similarity, Right Triangles, and Trigonometry

Understand similarity in terms of similarity transformations.

G-SRT.1 Verify experimentally the properties of dilations given by a center and a scale factor.

Apply geometric concepts in modeling situations

G-MG. 1. Use geometric shapes, their measures, and their properties to describe objects.

3. Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).

High School – Functions - Building Functions

Build a function that models a relationship between two quantities

F-BF.1. Write a function that describes a relationship between two quantities.

Construct and compare linear and exponential models and solve problems

F-LE.1. Distinguish between situations that can be modeled with linear functions and with exponential functions.

b. Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.

F-LE.2. Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).

Common Core State Standards Math – Standards of Mathematical Practice

MP.7 Look for and make use of structure.

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MP.8 Look for and express regularity in repeated reasoning.

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation (y – 2)/(x – 1) = 3. Noticing the regularity in the way terms cancel when expanding (x – 1)(x + 1), (x – 1)(x2 + x + 1), and (x – 1)(x3 + x2 + x + 1) might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Problem of the Month
Infinite Windows
Task Description – Primary Level
This task asks students to use spatial reasoning and geometric attributes to make sense of a figure. The students are asked to determine the number of rectangles in a figure comprised of three adjacent same-size rectangles. Their task is to identify the number of rectangles and show how they were counted.
Common Core State Standards Math - Content Standards
<u>**Geometry**</u> **Identify and describe shapes** K.G.2. Correctly name shapes regardless of their orientations or overall size. **Analyze, compare, create, and compose shapes.** K.G.4. Analyze and compare two- and three-dimensional shapes, in different sizes and orientations, using informal language to describe their similarities, differences, parts. K.G.5. Model shapes in the world by building shapes from components and drawing shapes. K.G.6. Compose simple shapes to form larger shapes. <u>**Counting and Cardinality K.CC**</u> **Count to tell the number of objects.** 5. Count to answer "how many?" questions about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; given a number from 1–20, count out that many objects.
Common Core State Standards Math – Standards of Mathematical Practice
MP.1 Make sense of problems and persevere in solving them. Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches. **MP.2 Reason abstractly and quantitatively.** Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects. **MP.7 Look for and make use of structure.** Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7 x 8 equals the well-remembered 7 x 5 + 7 x 3, in preparation for learning about the distributive property. In the expression $\mathbf{x}^2 + 9\mathbf{x} + 14$, older students can see the 14 as 2 x 7 and the 9 as 2 + 7. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(\mathbf{x} - \mathbf{y})^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers **x** and **y**.