Problem of the Month: *Fractured Numbers*

The Problems of the Month (POM) are used in a variety of ways to promote problem-solving and to foster the first standard of mathematical practice from the Common Core State Standards: "Make sense of problems and persevere in solving them." The POM may be used by a teacher to promote problem-solving and to address the differentiated needs of her students. A department or grade level may engage their students in a POM to showcase problem-solving as a key aspect of doing mathematics. It can also be used schoolwide to promote a problem-solving theme at a school. The goal is for all students to have the experience of attacking and solving non-routine problems and developing their mathematical reasoning skills. Although obtaining and justifying solutions to the problems is the objective, the process of learning to problem-solve is even more important.

The Problem of the Month is structured to provide reasonable tasks for all students in a school. The structure of a POM is a shallow floor and a high ceiling, so that all students can productively engage, struggle, and persevere. The Primary Version Level A is designed to be accessible to all students and especially the key challenge for grades K – 1. Level A will be challenging for most second and third graders. Level B may be the limit of where fourth and fifth grade students have success and understanding. Level C may stretch sixth and seventh grade students. Level D may challenge most eighth and ninth grade students, and Level E should be challenging for most high school students. These grade- level expectations are just estimates and should not be used as an absolute minimum expectation or maximum limitation for students. Problem-solving is a learned skill, and students may need many experiences to develop their reasoning skills, approaches, strategies, and the perseverance to be successful. The Problem of the Month builds on sequential levels of understanding. All students should experience Level A and then move through the tasks in order to go as deeply as they can into the problem. There will be those students who will not have access into even Level A. Educators should feel free to modify the task to allow access at some level.

Overview:

In the Problem of the Month *Fractured Numbers,* students explore rational numbers and solve problems involving symmetry, congruence, determining equal area, sub-dividing area models, reasoning about equivalent fractions, using operations of fractions to solve problems, investigating patterns involving fractions, and creating generalizations. The mathematical topics that underlie this POM include understanding rational numbers and operations through different situations and representations. Students explore fractions through area models, equivalence, division and rates, investigation of sums and infinite geometric series.

In the first level of the POM, students divide a geometric figure to determine the number pieces it takes to make a whole. Their task is to explore the pattern and relationship of halving half and the resulting fractions. In level B, students are given a fantasy situation to enter doors in a cave. The students are asked to find a way to categorize fractions that are equivalent. In level C, the students are presented with a situation that requires understanding of rates and the division of fractions. Students must solve problems involving feeding a dog different types and amounts of food. In level D, students investigate like fractions that sum to exactly 1. They are to determine which set of fractions with like denominators can sum to 1 and justify why only that set has that quality. In the final level of the POM, students are presented with a situation that invites an investigation of an infinite geometric series that converges. Students are asked to find generalizations and provide justifications.

Mathematical Concepts:
The mathematics involved in this Problem of the Month involves rational numbers. A rational number is a number that can be written as the quotient of two integers as long as the denominator is not zero. Rational numbers can be represented in numeric notation as fractions, decimals, and/or percents. Students can visualize rational numbers in area models, as points on number lines, and as a comparison between two discrete sets.

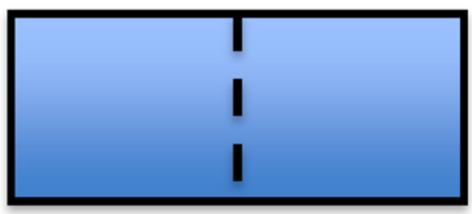
<u>**Problem of the Month**</u>

Fractured Numbers

Level A

Rosita has made a puzzle. She takes a whole rectangle like the one below.

She cuts the whole into half.



She takes that half and cuts it in half.

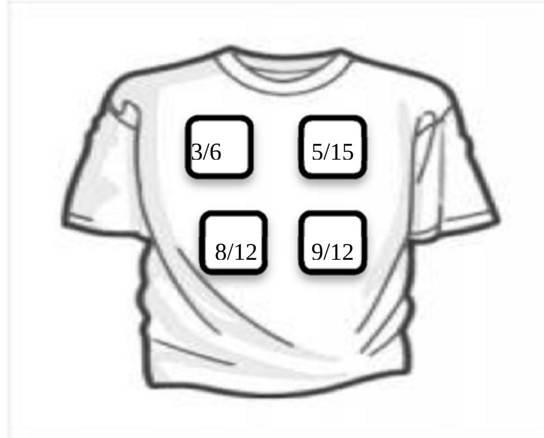
Finally she takes the small piece she cut and cuts that in half.

Now she has the piece she wants. How many of these small pieces can she put together to make the same size rectangle that she started with? Explain how you know.

If she had cut her small piece in half again, how many of those pieces would be needed to make the first rectangle?

What if she kept going? How would the pieces get smaller?

Level B



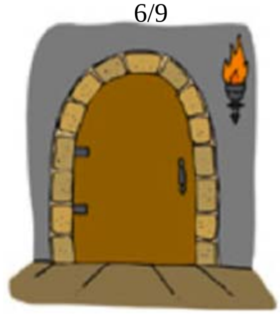
Jeff is playing a new video game. The goal is to explore a cave with many levels and collect gold coins. In order to get through a door to the next level, you must master a secret code. Jeff's video character has four pockets on his jacket. The top left pocket is marked 3/6, the top right pocket is marked 5/15, the lower left pocket is marked 8/12 and the lower right pocket is marked 9/12.

Above the cave door at each level is a number card. To open the door, he must take the number card from the door and put it in the correct pocket. If he puts it in the wrong pocket, he loses one of his 3 lives. Here are the cave doors he must travel through.

6/8

4/8

6/9



16/32

21/28

12/18

6/12

6/18

Figure out the secret code to get through all the doors. Write Jeff a note letting him know which numbers go into which pocket. Explain to Jeff how you know for sure. Remember, if you get it wrong he loses a life!

Level C

Roxie is a show dog. Her trainer wants her to have a beautiful and brilliant coat. The veterinarian suggested a special diet for the trainer to follow. Each feeding, Roxie eats 2/3 of a can of wet dog food, 1/8 of a bag of dry dog food, and 3/5 a patty of special meat. The special meat comes in a package of 6 patties. Roxie has two meals a day.

The dog is completely out of food. The trainer goes to the store and buys 24 cans of wet food, 4 bags of dry food and 3 packages of meat.

How many days will the dog be fed before the trainer needs to buy any more food?

Which type of dog food will the trainer run out of first? Explain.

How much of the other two types of dog food will be left after the first type of dog food runs out?

The trainer wants to plan better. She goes to the store on the day she ran out of the first type of dog food. She decides to buy enough dog food to last 90 days. Knowing what she already has in the house, how much more of each type of dog food does she need to buy in order to use up all the food in 90 days? Is it possible? Explain.

What is the minimum amount of food the trainer could buy such that the dog would finish all of it after a certain number of meals? Explain.

Level D

Tracy gave the following explanation to her friend.

I have found a special set of fractions that I call Consecutive Sum Fractions. To be a Consecutive Sum Fraction, it must be equivalent to 1. You start with the unit fraction and then add each larger fraction, the next natural number numerator of the same denominator, until you reach exactly one. For example, 15/15 is a Consecutive Sum Fraction, because;

1/15 + 2/15 + 3/15 + 4/15 + 5/15 = 1

Not all equivalent fractions equal to one are Consecutive Sum Fractions. For example consider 12/12. The follow partial sum approaches 1.

1/12 + 2/12 + 3/12 + 4/12 = 10/12

But it is still smaller than 1. If we add the next consecutive fraction, we get a fraction larger than 1.

1/12 + 2/12 + 3/12 + 4/12 + 5/12 = 15/12

Therefore, 12/12 is not a Consecutive Sum Fraction.

Which fractions are Consecutive Sum Fractions?

How can you determine or predict which will be Consecutive Sum Fractions?

Determine a means to generate all fractions that are Consecutive Sum Fractions.

Level E

Cheryl's mother baked a large rectangular pan of brownies for Cheryl and her two friends, Joanne and Richard, to share after school. Cheryl and Joanne got to the house at the same time. They knew Richard was coming over later.

They cut the brownie into three equal size pieces. Cheryl took one of the pieces and Joanne took another, leaving the third piece for Richard. Cheryl and Joanne each began to eat their share of brownies. When they were finished, Richard had still not shown up.

They were both still hungry so they decided to take Richard's piece and divide it into thirds again. Cheryl and Joanne began to eat their new "one-third" slices, leaving Richard with only one-third of his original piece.

Still Richard did not show, so the two friends decided to cut his remaining piece in thirds again. They set aside one of the cut pieces for Richard, and ate the other two.

If Richard never comes over to the house and the two friends continue their process of eating and dividing the remaining slices, how much will they each eat?

- Represent your answer in an equation.
- What can you conclude from an infinite sum?
- Suppose there were four friends and only three showed up to eat the brownies originally cut into fourths. If a similar process occurred how much would each of the three friends eat? Represent your conclusion in an equation with an infinite sum.
- Generalize your finding about similar sets of infinite sums.

2/3 **Problem of the Month** 3/4

Fractured Numbers

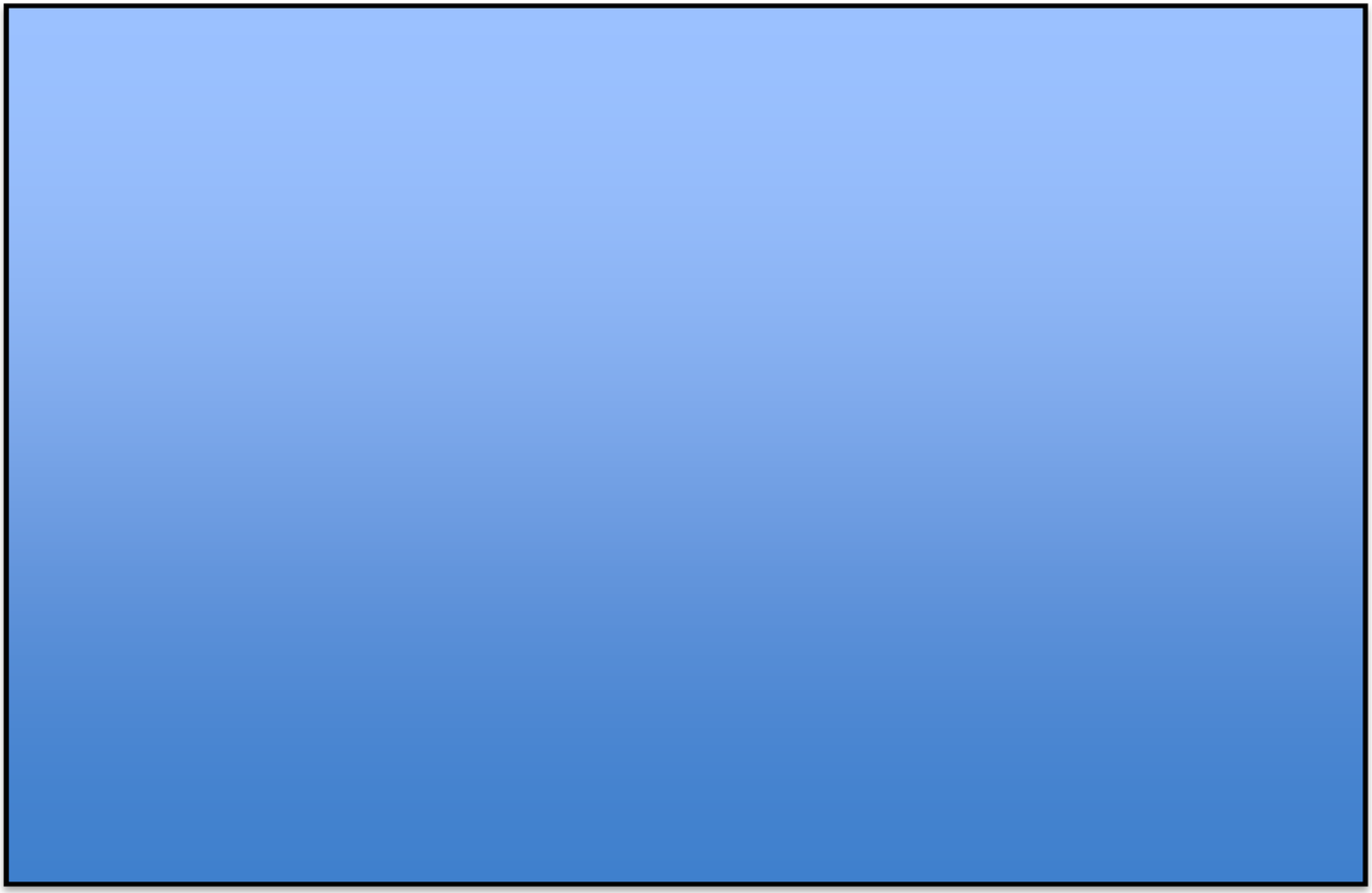
Primary Version Level A

Materials: For the teacher: A rectangular sheet of paper, a pair of scissors, several rectangles a fourth the size of the original, and a template of the rectangle. For each student: a template of the rectangle, baggie of lots of eighths of the rectangle, paper, and pencil.

Discussion on the rug: (Teacher shows a rectangle.) "What do we call this object?" (Students volunteer names. If students don't know, then the teacher states "We call it a rectangle.") "What do you notice about its shape?" (Students respond with ideas.) Teacher asks, "I am going to cut it into half. How many pieces will I have?" (Students respond and teacher cuts it and shows the results.) Teacher says, "I am going to take just one piece and cut it into half again. How big will the pieces be? How many of them fit on the first rectangle?" (Select a student to come forward and show the class that 4 pieces exactly fit onto the original rectangle.)

In small groups: (Each student has the materials)
The teacher says to the class, "Let's cut the small piece one more time. See how small it is now? How many pieces are needed to cover the rectangle now? Make a guess and then try it." (Students explore how many pieces [in eighths] are needed.)

(At the end of the investigation, have students either discuss or dictate a response to this summary question.) "How many pieces did you need? Tell me how you figured it out and how you know for sure."



Problem of the Month

Fractured Numbers

Task Description – Level A

This task asks students to explore the idea of a half, using symmetry. The students divide a geometric figure to determine the number pieces it takes to make a whole. Their task is to explore the pattern and relationship of halving half and the resulting fractions.

Common Core State Standards Math - Content Standards

Geometry

Reason with shapes and their attributes.

1.G.3. Partition circles and rectangles into two and four equal shares, describe the shares using the words halves, fourths, and quarters, and use the phrases half of, fourth of, and quarter of. Describe the whole as two of, or four of the shares. Understand for these examples that decomposing into more equal shares creates smaller shares.

2.G.3. Partition circles and rectangles into two, three, or four equal shares, describe the shares using the words halves, thirds, half of, a third of, etc., and describe the whole as two halves, three thirds, four fourths. Recognize that equal shares of identical wholes need not have the same shape.

Common Core State Standards Math – Standards of Mathematical Practice

MP.1 Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

MP.2 Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

Problem of the Month

Fractured Numbers

Task Description – Level B

This task challenges a student to recognize and make sense of equivalent fractions. The students are given a fantasy situation to enter doors in a cave. The students are asked to find a way to categorize fractions that are equivalent.

Common Core State Standards Math - Content Standards

<u>Number and Operations—Fractions</u>

Develop understanding of fractions as numbers.

3.NF.1. Understand a fraction 1/b as the quantity formed by 1 part when a whole is partitioned into b equal parts; understand a fraction a/b as the quantity formed by a parts of size 1/b.

3.NF.3. Explain equivalence of fractions in special cases, and compare fractions by reasoning about their size.

b. Recognize and generate simple equivalent fractions, e.g., 1/2 = 2/4, 4/6 = 2/3). Explain why the fractions are equivalent, e.g., by using a visual fraction model.

Extend understanding of fraction equivalence and ordering.

4.NF.1. Explain why a fraction **a**/**b** is equivalent to a fraction (**n** x **a**)/(**n** x **b**) by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions.

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Problem of the Month

Fractured Numbers

Task Description – Level C

This task challenges a student to make sense of a problem involving ratios, rates, and division. The students are presented with a situation that requires understanding of rates and the division of fractions. Students must solve problems involving feeding a dog different types and amounts of food.

Common Core State Standards Math - Content Standards

Number and Operations—Fractions

Use equivalent fractions as a strategy to add and subtract fractions.

5.NF.2. Solve word problems involving addition and subtraction of fractions referring to the same whole, including cases of unlike denominators.

Apply and extend previous understandings of multiplication and division to multiply and divide fractions.

5.NF.3. Interpret a fraction as division of the numerator by the denominator ($a/b = a \div b$). Solve word problems involving division of whole numbers leading to answers in the form of fractions or mixed numbers

5.NF.7. Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions.

c. Solve real world problems involving division of unit fractions by non-zero whole numbers and division of whole numbers by unit fractions.

Ratios and Proportional Relationships

Understand ratio concepts and use ratio reasoning to solve problems.

6.RP.3. Use ratio and rate reasoning to solve real-world and mathematical problems,

The Number System

Apply and extend previous understandings of multiplication and division to divide fractions by fractions.

6.NS.1. Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions

Common Core State Standards Math – Standards of Mathematical Practice

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MP.2 Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

Problem of the Month:

Fractured Numbers

Task Description – Level D

This task challenges a student to explore a number theory problem involving fractions. The students investigate like denominator fractions that sum to exactly 1. They are to determine which set of fractions with like denominators can sum to 1 and justify why only that set has that quality.

Common Core State Standards Math - Content Standards

<u>The Number System</u>

Apply and extend previous understandings of operations with fractions to add, subtract, multiply, and divide rational numbers.

7.NS.1. Apply and extend previous understandings of addition and subtraction to add and subtract rational numbers; represent addition and subtraction on a horizontal or vertical number line diagram.

Common Core State Standards Math – Standards of Mathematical Practice

MP.2 Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

MP.7 Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7 x 8 equals the well-remembered 7 x 5 + 7 x 3, in preparation for learning about the distributive property. In the expression $\mathbf{x}^2 + 9\mathbf{x} + 14$, older students can see the 14 as 2 x 7 and the 9 as 2 + 7. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 – 3(\mathbf{x} – \mathbf{y})^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers **x** and **y**.

MP.8 Look for and express regularity in repeated reasoning.

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation (y – 2)/(x – 1) = 3. Noticing the regularity in the way terms cancel when expanding (x – 1)(x + 1), (x – 1)(x2 + x + 1), and (x – 1)(x3 + x2 + x + 1) might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Problem of the Month

Fractured Numbers

Task Description – Level E

This task challenges a student to explore a number theory problem involving rational numbers. The students are presented with a situation that invites an investigation of infinite geometric series that converges. Students are asked to find generalizations and provide justifications.

Common Core State Standards Math - Content Standards

High School – Functions - Interpreting Functions

Understand the concept of a function and use function notation

F-IF.3. Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers.

Interpret functions that arise in applications in terms of the context

F-IF.4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

High School – Functions - Building Functions

Build a function that models a relationship between two quantities

F-BF.2. Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.

High School – Functions - Linear and Exponential Models

Construct and compare linear and exponential models and solve problems.

F-LE.2. Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs.

Common Core State Standards Math – Standards of Mathematical Practice

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Problem of the Month
Fractured Numbers
Task Description – Primary Level
This task asks students to explore the idea of cutting pieces of paper in halves and counting the number of pieces. The students divide a plain sheet of paper in half to determine the number of pieces it takes to make a whole. They then cut one of the two pieces in half. That process is repeated. The students' task is to determine the number of pieces that were created by cutting the paper.
Common Core State Standards Math - Content Standards
Counting and Cardinality **Count to tell the number of objects.** K.CC.4. Understand the relationship between numbers and quantities; connect counting to cardinality. K.CC.5. Count to answer "how many?" questions about as many as 20 things arranged in a line, a rectangular array, or a circle, or as many as 10 things in a scattered configuration; given a number from 1–20, count out that many objects **Geometry** **Identify and describe shapes** K.G.1. Describe objects in the environment using names of shapes, and describe the relative positions of these objects using terms such as above, below, beside, in front of, behind, and next to. K.G.2. Correctly name shapes regardless of their orientations or overall size. Analyze, compare, create, and compose shapes. K.G.4. Analyze and compare two- and three-dimensional shapes, in different sizes and orientations, using informal language to describe their similarities, differences, parts.
Common Core State Standards Math – Standards of Mathematical Practice
MP.2 Reason abstractly and quantitatively. Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects. **MP.7 Look for and make use of structure.** Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7 x 8 equals the well-remembered 7 x 5 + 7 x 3, in preparation for learning about the distributive property. In the expression $\mathbf{x}^2 + 9\mathbf{x} + 14$, older students can see the 14 as 2 x 7 and the 9 as 2 + 7. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(\mathbf{x} - \mathbf{y})^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers **x** and **y**.