Problem of the Month: *Calculating Palindromes*

The Problems of the Month (POM) are used in a variety of ways to promote problem solving and to foster the first standard of mathematical practice from the Common Core State Standards: "Make sense of problems and persevere in solving them." The POM may be used by a teacher to promote problem solving and to address the differentiated needs of her students. A department or grade level may engage their students in a POM to showcase problem solving as a key aspect of doing mathematics. POMs can also be used school wide to promote a problem-solving theme at a school. The goal is for all students to have the experience of attacking and solving non-routine problems and developing their mathematical reasoning skills. Although obtaining and justifying solutions to the problems is the objective, the process of learning to problem solve is even more important.

The Problem of the Month is structured to provide reasonable tasks for all students in a school. The structure of a POM is a shallow floor and a high ceiling, so that all students can productively engage, struggle, and persevere. The Primary Version is designed to be accessible to all students and especially the key challenge for grades K – 1. Level A will be challenging for most second and third graders. Level B may be the limit of where fourth and fifth grade students have success and understanding. Level C may stretch sixth and seventh grade students. Level D may challenge most eighth and ninth grade students, and Level E should be challenging for most high school students. These grade- level expectations are just estimates and should not be used as an absolute minimum expectation or maximum limitation for students. Problem-solving is a learned skill, and students may need many experiences to develop their reasoning skills, approaches, strategies, and the perseverance to be successful. The Problem of the Month builds on sequential levels of understanding. All students should experience Level A and then move through the tasks in order to go as deeply as they can into the problem. There will be those students who will not have access into even Level A. Educators should feel free to modify the task to allow access at some level.

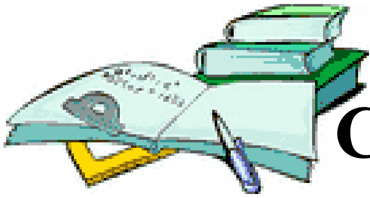
Overview

In the Problem of the Month, *Calculating Palindromes,* students investigate classic number theory problems. They explore number operations, organized lists and counting methods to solve problems. The mathematical topics that underlie this POM are knowledge of number sense, addition, subtraction, division, factors and divisibility, counting principles, systematic charting and closed-form equations. The mathematics that includes counting principles and systemic charting is often referred to as discrete mathematics. The higher levels of the POM involve students investigating two classic number theory problems, the Kaprekar series and the consecutive sums problem. These problems involve place value, divisibility, factors, primes and composites.

In the first level of the POM, students are asked to find and list all the ways two counting numbers are added to equal 12. Their task involves making systematic lists. The second part is to find the ways to sum 3 counting numbers that equal 12. In level B, students examine a classic process. It involves finding the difference between the largest and smallest number created from a set of two different digits. The students find the result of the subtraction problem, and then use those digits to repeat the process. Students track and determine patterns that occur from the set of differences. They explain the pattern they found and discuss how those results are related. In level C, students are asked to carry out a similar investigation to level B but with three digits. In level D, the students investigate that same process with 4 digit numbers. In level E, students are asked to find and investigate all the ways to find integers using the sum of consecutive positive integers.

Mathematical Concepts

The big idea of this POM is number theory – the investigation of our real number system and its operations. Number theory is one of the oldest branches of pure mathematics, and one of the largest. Of course, it concerns questions about numbers, usually meaning whole numbers or rational numbers (fractions). Elementary number theory involves divisibility among integers -- the Euclidean algorithm (and thus the existence of greatest common divisors), elementary properties of primes (the unique factorization theorem, the infinitude of primes), etc. Tools for solving some of the number theory questions draw ideas from discrete mathematics. Discrete mathematics is the study of the sets of and operations on discrete objects. Discrete means distinct from others, separate or discontinuous. Basically discrete is not continuous mathematics. Traditionally, K-12 mathematics starts with discrete topics in lower grades, but by secondary mathematics the topics usually focus on continuous mathematics as a preparation for calculus. The real number line is an example of a continuous set of points. But, discrete mathematics plays an important role in higher mathematics courses. Much of the mathematics that comes from real life comes in the form of distinct objects and values. Making sense of this distinct information and quantities is important; thus discrete mathematics has numerous applications to real life.



Problem of the Month

Calculating Palindromes

Level A

Find and list all the ways you can add two counting numbers to equal 12. The order that numbers are added doesn't affect whether two number sentences are the same.

What patterns do you see in the number sentences?

How do you know you have found all possible number sentences? Explain.

Find and list all the ways you can add three counting numbers to equal 12.

Level B

Pick any 2-digit number where not all the digits are equal. Order the digits from highest to lowest to create the largest number. Next order the digits from lowest to highest to create the smallest number. Find the positive difference between the two numbers. Investigate different solutions you find. Are there patterns? If so, what patterns did you find?

Level C

Pick any 3-digit number where not all the digits are equal. Order the digits from highest to lowest to create the largest number. Next order the digits from lowest to highest to create the smallest number. Find the positive difference between the two numbers. Investigate different solutions you find. Are there patterns? If so, what patterns did you find?

Do two different 3–digit numbers produce the same solution (difference) when following the process?

What can you predict about the solutions (differences) in terms of specific digits (hundreds, tens, ones)?

What other relationships are there between the digits?

Explain your findings

Level D
Pick any 4-digit number where not all the digits are equal. Order the digits from highest to lowest to create the largest number X. Next order the digits from lowest to highest to create the smallest number Y. Subtract to form X-Y. Repeat the process with X-Y. Continue.

Following the process, eventually you will come to a terminating number. What is the terminating number? What makes it terminate?

Can you predict the number of steps to the terminating number?

What is the maximum number of steps?

What are some patterns you notice in the numbers formed along the way to the terminating number?

Compare what happens for 3-digit numbers? 2-digit numbers? 5-digit numbers?

Level E

Numbers may be expressed as the sum of consecutive positive integers.

For example:

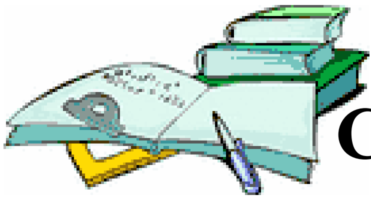
$12 = 3 + 4 + 5$
$13 = 6 + 7$
$25 = 3 + 4 + 5 + 6 + 7$

Which numbers can be expressed as consecutive positive integers? How do you know you have found them all? Explain.

Which numbers can be expressed as sums of two consecutive integers? Three?

How many different ways are there of expressing numbers as sums of consecutive integers? Justify your findings.

Investigate numbers that can be expressed as sums of consecutive odd integers. Explain your findings.



Problem of the Month
Calculating Palindromes

Primary Version Level A

Materials: A set of blocks or counters for each pair, pencil and paper.

Discussion on the rug: Teacher shows the pattern blocks. **"How can we add to 4? Who can tell me a number sentence that adds up to 4?"** Teacher asks a student and records the students' number sentences. **"Who knows a different way to add to 4?"** Teacher continues to ask children to find different number sentences. **"Have we found every way? How will we know when we are done?"** Students share their thoughts.

In small groups: Each group has a set of blocks or counters.
"How many ways can you add to get a total of 8? Show me all your number sentences."

At the end of the investigation have students write all their number sentences. Have them answer the following questions:

"How did you find all the ways to add to 8?"

"Do you think you found every way?"

"How many ways are there?"

"Why do you think you know you found them all?"

Problem of the Month
Calculating Palindromes
Task Description – Level A
This task challenges students to use number concepts. The students are asked to find and list all the ways two counting numbers are added to equal 12. Their task involves making systematic lists. The second part is to find the ways to sum 3 counting numbers that equal 12.
Common Core State Standards Math - Content Standards
Operations and Algebraic Thinking **Represent and solve problems involving addition and subtraction.** 1.OA.1. Use addition and subtraction within 20 to solve word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem. 1.OA.2. Solve word problems that call for addition of three whole numbers whose sum is less than or equal to 20, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem. **Understand and apply properties of operations and the relationship between addition and subtraction.** 1.OA.3. Apply properties of operations as strategies to add and subtract. Examples: If 8 + 3 = 11 is known, then 3 + 8 = 11 is also known. (Commutative property of addition.) To add 2 + 6 + 4, the second two numbers can be added to make a ten, so 2 + 6 + 4 = 2 + 10 = 12. (Associative property of addition.)
Common Core State Standards Math – Standards of Mathematical Practice
MP.1 Make sense of problems and persevere in solving them. Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, “Does this make sense?” They can understand the approaches of others to solving complex problems and identify correspondences between different approaches. **MP.2 Reason abstractly and quantitatively**. Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

Problem of the Month
Calculating Palindromes
Task Description – Level B
This task challenges students to use number concepts. The students examine a classic process. It involves finding the difference between the largest and smallest number created from a set of two different digits. The students find the result of the subtraction problem, and then use those digits to repeat the process. Students track and determine patterns that occur from the set of differences. They explain the pattern they found and discuss how those results are related.
Common Core State Standards Math - Content Standards
<u>**Operations and Algebraic Thinking**</u> **Represent and solve problems involving addition and subtraction.** 2.OA.1. Use addition and subtraction within 100 to solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem**.** **Operations and Algebraic Thinking 3.OA** **Solve problems involving the four operations, and identify and explain patterns in arithmetic**. 8. Solve two-step word problems using the four operations. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding.
Common Core State Standards Math – Standards of Mathematical Practice
MP.2 Reason abstractly and quantitatively. Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects. **MP.7 Look for and make use of structure.** Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7 x 8 equals the well-remembered 7 x 5 + 7 x 3, in preparation for learning about the distributive property. In the expression $\mathbf{x}^2 + 9\mathbf{x} + 14$, older students can see the 14 as 2 x 7 and the 9 as 2 + 7. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(\mathbf{x} - \mathbf{y})^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers **x** and **y**.

Problem of the Month
Calculating Palindromes
Task Description – Level C
This task challenges a student to investigate a problem involving number patterns and number theory. The students are asked to carry out a similar investigation to level B but with three digits. The students find the result of the subtraction problem, and then use those digits to repeat the process. Students track and determine patterns that occur from the set of differences. They explain the pattern they found and discuss how those results are related.
Common Core State Standards Math - Content Standards
<u>**Expressions and Equations**</u> **Represent and analyze quantitative relationships between dependent and independent variables.** 6.EE.9. Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation. **Use properties of operations to generate equivalent expressions**. 7.EE.1. Apply properties of operations as strategies to add, subtract, factor, and expand linear expressions with rational coefficients. **Solve real-life and mathematical problems using numerical and algebraic expressions and equations.** 7.EE.4. Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems by reasoning about the quantities.
Common Core State Standards Math – Standards of Mathematical Practice
MP.1 Make sense of problems and persevere in solving them. Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches. **MP.4 Model with mathematics.** Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

Problem of the Month:

Calculating Palindromes

Task Description – Level D

This task challenges a student to investigate a problem involving number patterns and number theory. The students are asked to carry out a similar investigation to level B and C but with four digits. The students find the result of the subtraction problem, and then use those digits to repeat the process. Students track and determine patterns that occur from the set of differences. They explain the pattern they found and discuss how those results are related.

Common Core State Standards Math - Content Standards

Operations and Algebra

Gain familiarity with factors and multiples.

4.OA.4 Find all factor pairs for a whole number in the range 1-100. Recognize that a whole number is a multiple of each of the factors. Determine whether a given whole number in the range 1-100 is a multiple of a given one-digit number. Determine whether a given whole number in the range 1-100 is a prime or composite.

Expressions and Equations

Represent and analyze quantitative relationships between dependent and independent variables.

6.EE.9 Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variable using graphs and tables, and relate these to the equations.

Solve real-life and mathematical problems using numerical and algebraic expressions and equations.

7.EE.4 Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems by reasoning about the quantities.

Common Core State Standards Math – Standards of Mathematical Practice

MP.4 Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

MP.7 Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7 x 8 equals the well-remembered 7 x 5 + 7 x 3, in preparation for learning about the distributive property. In the expression $\mathbf{x}^2 + 9\mathbf{x} + 14$, older students can see the 14 as 2 x 7 and the 9 as 2 + 7. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(\mathbf{x} - \mathbf{y})^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers **x** and **y**.

Problem of the Month

Calculating Palindromes

Task Description – Level E

This task challenges a student to investigate problems involving number theory. The students are asked to find and investigate all the ways to find integers using the sum of consecutive positive integers.

Common Core State Standards Math - Content Standards

High School – Functions – Building Functions

Build a function that models a relationship between two quantities.

F-BF.1 Write a function that describes a relationship between two quantities.

Interpret expressions for functions in terms of the situation they model.

F-LE.5 Interpret the parameters in a linear or exponential function in terms of a context.

High School – Algebra - Creating Equations

Create equations that describe numbers or relationships.

A-CED.2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

Common Core State Standards Math – Standards of Mathematical Practice

MP.7 Look for and make use of structure.

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MP.8 Look for and express regularity in repeated reasoning.

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1,2) with slope 3, middle school students might abstract the equation (y -2)/(x-1) = 3. Noticing the regularity in the way terms cancel when expanding (x-1)(x+1), $(x-1)(x^2 + x + 1)$, and $(x-1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Problem of the Month

Calculating Palindromes

Task Description – Primary Level

This task challenges students to use number concepts to solve problems. The students are asked to find all the natural numbers that add to make a total of 8. The students explain how they found all their solutions and why they think they found all possible number sentences.

Common Core State Standards Math - Content Standards

Operations and Algebraic Thinking

Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from.

K.OA.1. Represent addition and subtraction with objects, fingers, mental images, drawings, sounds (e.g., claps), acting out situations, verbal explanations, expressions, or equations.

K.OA.2. Solve addition and subtraction word problems, and add and subtract within 10, e.g., by using objects or drawings to represent the problem.

K.OA.3. Decompose numbers less than or equal to 10 into pairs in more than one way, e.g., by using objects or drawings, and record each decomposition by a drawing or equation (e.g., 5 = 2 + 3 and 5 = 4 + 1).

Common Core State Standards Math – Standards of Mathematical Practice

MP.1 Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

MP.2 Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.