Problem of the Month: *Perfect Pair*

The Problems of the Month (POM) are used in a variety of ways to promote problem-solving and to foster the first standard of mathematical practice from the Common Core State Standards: "Make sense of problems and persevere in solving them." The POM may be used by a teacher to promote problem-solving and to address the differentiated needs of her students. A department or grade level may engage their students in a POM to showcase problem-solving as a key aspect of doing mathematics. It can also be used schoolwide to promote a problem-solving theme at a school. The goal is for all students to have the experience of attacking and solving non-routine problems and developing their mathematical reasoning skills. Although obtaining and justifying solutions to the problems is the objective, the process of learning to problem-solve is even more important.

The Problem of the Month is structured to provide reasonable tasks for all students in a school. The structure of a POM is a shallow floor and a high ceiling, so that all students can productively engage, struggle, and persevere. The Primary Version Level A is designed to be accessible to all students and especially the key challenge for grades K – 1. Level A will be challenging for most second and third graders. Level B may be the limit of where fourth and fifth grade students have success and understanding. Level C may stretch sixth and seventh grade students. Level D may challenge most eighth and ninth grade students, and Level E should be challenging for most high school students. These grade- level expectations are just estimates and should not be used as an absolute minimum expectation or maximum limitation for students. Problem-solving is a learned skill, and students may need many experiences to develop their reasoning skills, approaches, strategies, and the perseverance to be successful. The Problem of the Month builds on sequential levels of understanding. All students should experience Level A and then move through the tasks in order to go as deeply as they can into the problem. There will be those students who will not have access into even Level A. Educators should feel free to modify the task to allow access at some level.

Overview:

In the Problem of the Month *Perfect Pair*, students are engaged in tasks and puzzles that involve properties, equality, equations, and variables. The mathematical topics that underlie this POM are number properties, operations, number theory, equality, variables, identities, inverse operations, solving equations, generalizing, and algebraic thinking.

In the first levels of the POM, students are presented with a task involving number sentences involving addition and subtraction using the same two numbers. The task involves finding the addition identity. In level B, students are presented with equations and unknown symbolized as different types of shoes. They need to determine which digit (0-9) corresponds to each style shoe. Students need to use their knowledge of the numbers and operations to deduce and solve the unknowns. In level C, students are presented with two different relationships of three unknowns and different operations. The students are asked to determine the sets of values for the unknowns that make the relationship equal. In level D, students are given a similar situation to the one they had in level B, but this time they have to determine values for when the sum of two numbers equals the product of the two numbers. The goal is to find the set of numbers that satisfy that relationship. In the final level E, students are asked to consider when the sum of two numbers can equal the quotient of the same two numbers. The task is to find the generalization for all number pairs that satisfy the situation and characterize the set.

Mathematical Concepts:

Algebra is the cornerstone of secondary mathematics. Algebraic thinking is taught in primary grades with the foundations of algebra taught usually by the end of middle school. Even though the term "algebraic thinking" is routinely used, it can not be simply defined. The underpinnings of algebra involve abstractions and language. There are several resources that define the most important concepts in algebra. One resource is *Fostering Algebraic Thinking,* by Mark Driscoll, published by Heinemann, 1999.

Problem of the Month

Perfect Pair

Level A

When you add two numbers you get a certain answer. Using the same two numbers, subtract the smaller from the larger number. If the two answers are the same, we will call that a perfect pair.

Can you find two numbers that are a perfect pair?

If you think it is impossible, explain why.

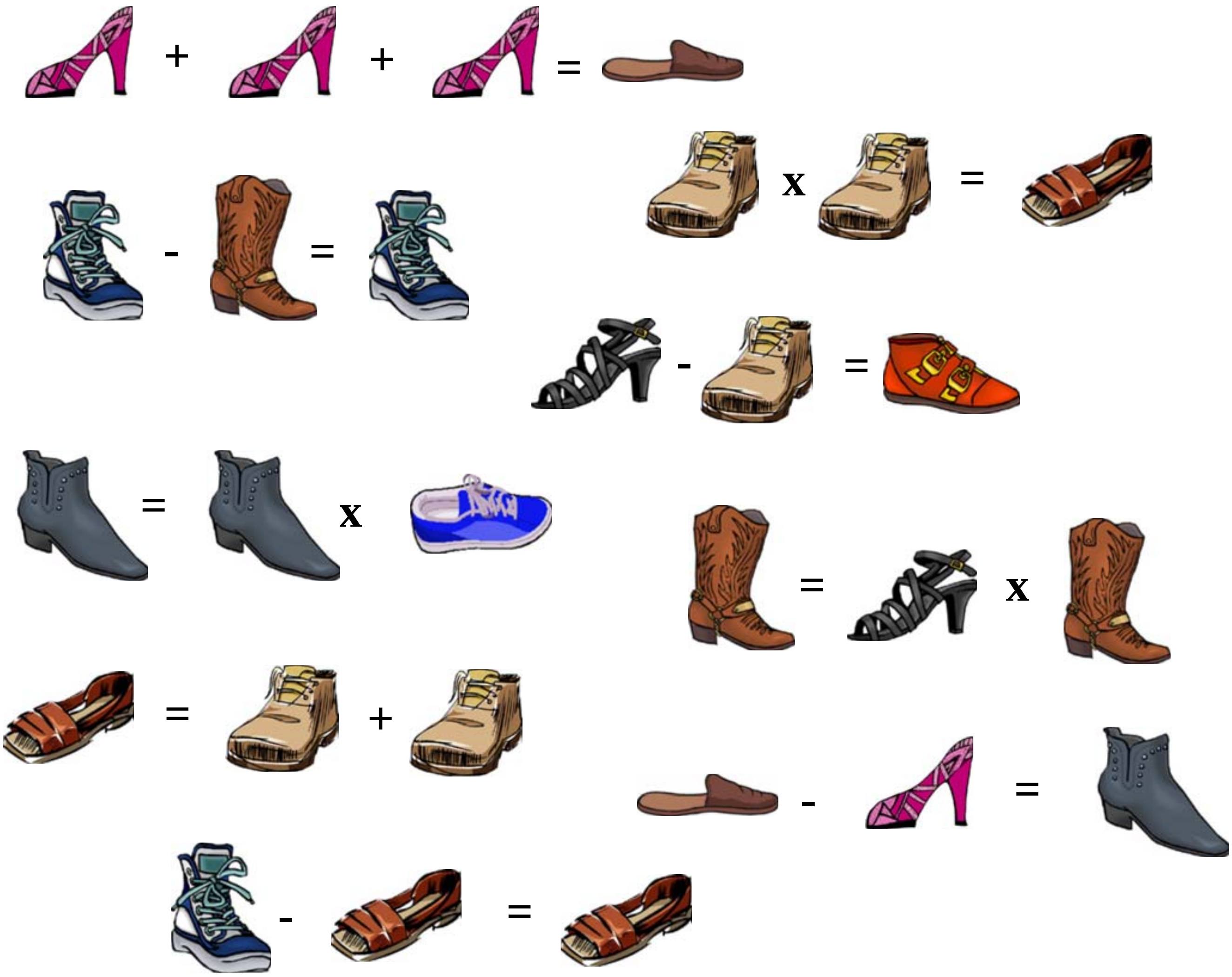
If you have found a perfect pair, explain why they are a perfect pair.

If it is possible, is there more than one perfect pair? How many are there?

Explain all you know about perfect pairs and what is special about them.

Level B

Each shoe represents a digit (0, 1, 2, 3, 4, 5, 6, 7, 8, 9). The style of shoe always represents the same digit in all of the number sentences. Examine the number sentences.



Pair each shoe with a digit. Explain your reasoning.

Level C

A perfect trio involves three whole numbers. Using the three numbers, add the first two numbers together then divide the sum by the third number. Using the same three numbers, subtract the second from the first number and then multiple the difference by the third. The trio is perfect if the two outcomes are equal.

Can you find three whole numbers that are perfect trios?

If not, how can you show that there are not any perfect trios? Is there a way to prove there are no perfect trios? Explain completely.

If you can find a perfect trio, is there more than one perfect trio? If so, how many? List the trio(s) you found. How do you know if you found them all? Describe any special characteristics of perfect trios. How can you go about finding them? Explain completely.

Level D

In this problem, a perfect pair is defined as two numbers whose sum is equal to their product.

Explore these perfect pairs.

If you cannot find any perfect pairs, prove that a perfect pair cannot exist.

If you find perfect pairs, then generalize your findings and describe the relationship of the number pairs. Illustrate the set of perfect pairs using multiple representations (words, symbols, graphs, tables, diagrams).

Level E

In this problem, a perfect pair is defined as two numbers whose sum is equal to their quotient.

Explore these perfect pairs.

If you cannot find any perfect pairs, prove that a perfect pair cannot exist.

If you find perfect pairs, then generalize your findings and describe the relationship of the number pairs. Illustrate the set of perfect pairs using multiple representations (words, symbols, graphs, tables, diagrams).

Problem of the Month

Perfect Pair

Primary Version Level A

Materials: Number cards 0 – 9, a paper bag, counters, paper, and pencil.

Discussion on the rug: Teacher says to the class, "I have numbers in my bag." The teacher shows the class the contents of the bag and shows the cards with single digits, counting them from 0 to 5. The teacher asks a student to pick out a number from the bag and show it to the class. The teachers says, "What is the name of that number? Show on your fingers how many it is." The class responds. The teacher then asks a second person to pick a different number from the bag and shows it to the class. The teacher says, "If we added the two numbers, how many would we have?" The teacher asks for answers and then asks students to prove it using counters, or drawings, or demonstrate with fingers, etc. (The process can be repeated until the students have comfort with the procedure of adding on.) The teacher now demonstrates an activity that requires students to find the missing addend. First the teacher picks a number out of the bag and shows the class. The teacher pulls a 1 card from the bag but turns it face down. The teacher mentally calculates and writes the sum on the board and states, "What was the second number I would have had to pick to get this answer?" The teacher asks the students to respond and justify their answer.

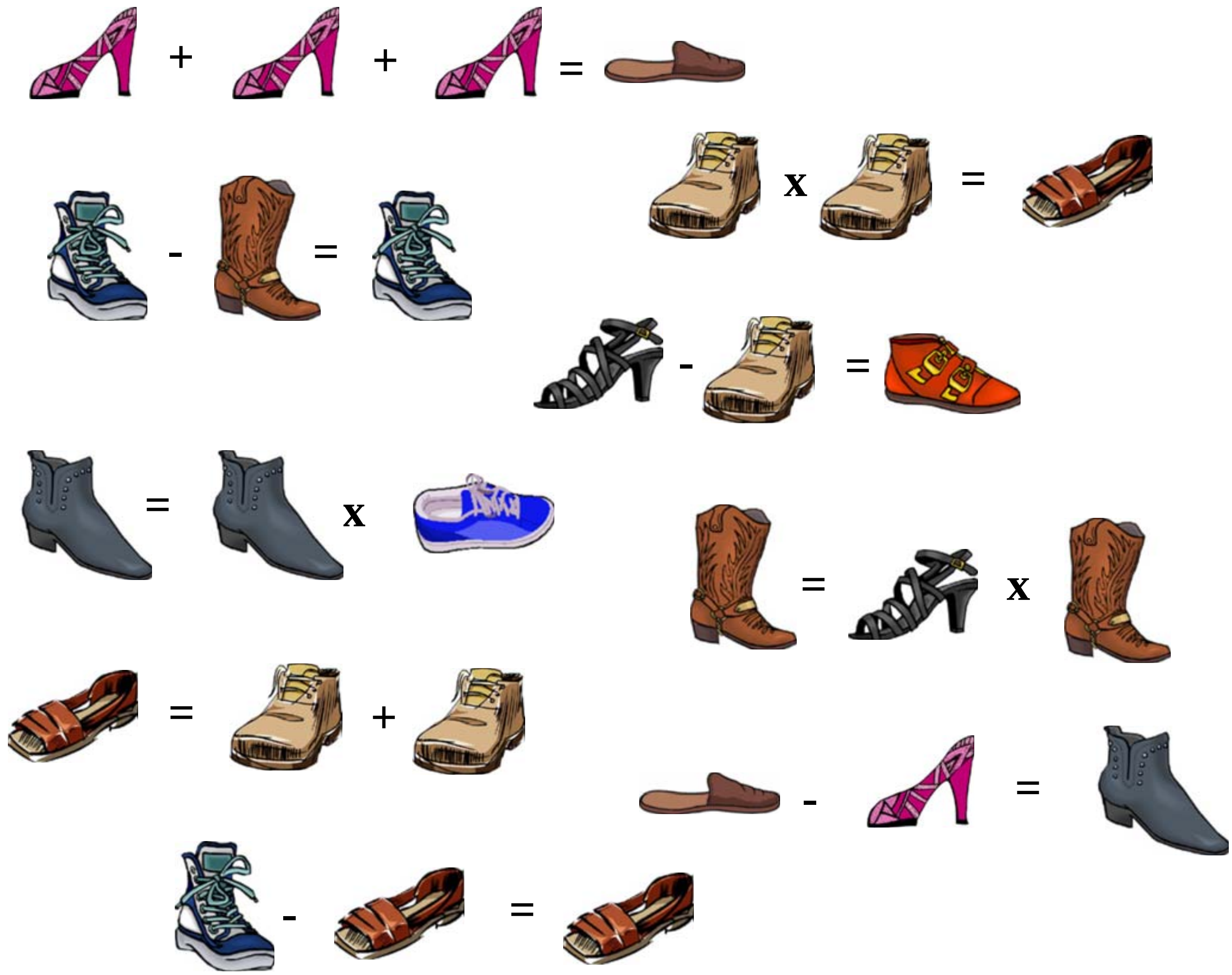
Back at their desks: Each student has access to a bag with numbers 0 - 9, counters, pencil, and paper. The teacher picks a number from the bag. The teacher picks a second number, hiding it and writes the sum on the board. The students are given time to calculate the missing addend using a process that makes sense to them. This game is repeated several times, including a time when zero is used as the first number drawn and also as the second number drawn. At the end of the investigation, have students either discuss or dictate a response to the prompt: "How did you find the hidden number? Tell me how to find it no matter which numbers are picked."

0	0	0	0
0	0	0	0
0	0	0	0

1	1	1	1
2	2	2	2
3	3	3	3

4	4	4	4
5	5	5	5
6	6	6	6

7	7	7	7
8	8	8	8
9	9	9	9



Pair each shoe with a digit. Explain your reasoning.

Problem of the Month

Perfect Pair

Task Description – Level A

This task challenges students to use algebraic thinking and operations to solve problems. Students are given a set of statements and asked to find a set of two numbers that makes the statements true. Students create mathematical models and explain their thinking.

Common Core State Standards Math - Content Standards

Operations and Algebraic Thinking

Represent and solve problems involving addition and subtraction.

1.OA.1. Use addition and subtraction within 20 to solve word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.2

1.OA.2. Solve word problems that call for addition of three whole numbers whose sum is less than or equal to 20, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem.

Understand and apply properties of operations and the relationship between addition and subtraction.

1.OA.3. Apply properties of operations as strategies to add and subtract.3 Examples:
If 8 + 3 = 11 is known, then 3 + 8 = 11 is also known. (Commutative property of addition.) To add 2 + 6 + 4, the second two numbers can be added to make a ten, so 2 + 6 + 4 = 2 + 10 = 12. (Associative property of addition.)

Work with addition and subtraction equations.

1.OA.7. Understand the meaning of the equal sign, and determine if equations involving addition and subtraction are true or false. For example, which of the following equations are true and which are false? 6 = 6, 7 = 8 – 1, 5 + 2 = 2 + 5, 4 + 1 = 5 + 2.

1.OA.8. Determine the unknown whole number in an addition or subtraction equation relating to three whole numbers. For example, determine the unknown number that makes the equation true in each of the equations 8 +? = 11, 5 = □ – 3, 6 + 6 = □.

Common Core State Standards Math – Standards of Mathematical Practice

MP.4 Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

MP.7 Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7 x 8 equals the well-remembered 7 x 5 + 7 x 3, in preparation for learning about the distributive property. In the expression $\mathbf{x}^2 + 9\mathbf{x} + 14$, older students can see the 14 as 2 x 7 and the 9 as 2 + 7. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(\mathbf{x} - \mathbf{y})^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers **x** and **y**.

Problem of the Month

Perfect Pair

Task Description – Level B

This task challenges a student to use understanding of number and operation in order to solve a set of problems involving unknowns. The students are given ten different styles of shoes that are used in nine equations. The students reason about the relationship of the numbers and operations to determine which single-digit number is represented by the style of shoe.

Common Core State Standards Math - Content Standards

Operations and Algebraic Thinking

Represent and solve problems involving addition and subtraction.

2.OA.1. Use addition and subtraction within 100 to solve one- and two-step word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using drawings and equations with a symbol for the unknown number to represent the problem**.**

Represent and solve problems involving multiplication and division.

3.OA.4. Determine the unknown whole number in a multiplication or division equation relating three whole numbers

Understand properties of multiplication and the relationship between multiplication and division.

3.OA.5. Apply properties of operations as strategies to multiply and divide.

3.OA.6. Understand division as an unknown-factor problem.

Common Core State Standards Math – Standards of Mathematical Practice

MP.2 Reason abstractly and quantitatively.

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

MP.7 Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7 x 8 equals the well-remembered 7 x 5 + 7 x 3, in preparation for learning about the distributive property. In the expression $\mathbf{x}^2 + 9\mathbf{x} + 14$, older students can see the 14 as 2 x 7 and the 9 as 2 + 7. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(\mathbf{x} - \mathbf{y})^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers **x** and **y**.

Problem of the Month
Perfect Pair
Task Description – Level C
This task challenges a student to find three numbers that fit a set of constraints. A student is to find three numbers to meet the conditions of adding the first two numbers together then dividing the sum by the third number and also subtracting the second from the first number and then multiplying the difference by the third number. The students must explain why finding these three numbers is or is not possible and whether there is more than one set of numbers that meets these conditions.
Common Core State Standards Math - Content Standards
<u>**Operations and Algebraic Thinking**</u> **Use the four operations with whole numbers to solve problems.** 4.OA.3. Solve multistep word problems posed with whole numbers and having whole-number answers using the four operations, including problems in which remainders must be interpreted. Represent these problems using equations with a letter standing for the unknown quantity. Assess the reasonableness of answers using mental computation and estimation strategies including rounding. **Write and interpret numerical expressions.** 5.OA.2. Write simple expressions that record calculations with numbers, and interpret numerical expressions without evaluating them.
Common Core State Standards Math – Standards of Mathematical Practice
MP.3 Construct viable arguments and critique the reasoning of others. Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments. **MP.4 Model with mathematics.** Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

Problem of the Month:
Perfect Pair
Task Description – Level D
This task challenges a student to investigate a number theory problem and justify whether or not there exists a pair of numbers that meets the constraints. The constraints are two numbers whose sum is equal to their product. The student must either prove that such a set does not exist or create a generalization for all pairs of numbers that meet those conditions.
Common Core State Standards Math - Content Standards
<u>**Expressions and Equations**</u> **Apply and extend previous understandings of arithmetic to algebraic expressions.** 6.EE.2. Write, read, and evaluate expressions in which letters stand for numbers. a. Write expressions that record operations with numbers and with letters standing for numbers. **Represent and analyze quantitative relationships between dependent and independent variables.** 6.EE.9. Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation. **Use properties of operations to generate equivalent expressions**. 7.EE.1. Apply properties of operations as strategies to add, subtract, factor, and expand linear expressions with rational coefficients. **Solve real-life and mathematical problems using numerical and algebraic expressions and equations** 7.EE.4. Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems by reasoning about the quantities.
Common Core State Standards Math – Standards of Mathematical Practice
MP.3 Construct viable arguments and critique the reasoning of others. Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments. **MP.4 Model with mathematics.** Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

Problem of the Month

Perfect Pair

Task Description – Level E

This task challenges a student to investigate a number theory problem and justify whether or not there exists a pair of numbers that meets the constraints. The constraints are two numbers whose sum is equal to their quotient. The students must either prove that such a set does not exist or create a generalization for all pairs of numbers that meet those conditions.

Common Core State Standards Math - Content Standards

High School – Algebra - Seeing Structure in Expressions
Write expressions in equivalent forms to solve problems
A-SSE.3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.

High School – Algebra - Creating Equations
Create equations that describe numbers or relationships
A-CED.2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

High School – Algebra - Reasoning with Equations and Inequalities
Represent and solve equations and inequalities graphically
A-REI.10. Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).

Common Core State Standards Math – Standards of Mathematical Practice

MP.2 Reason abstractly and quantitatively.
Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

MP.7 Look for and make use of structure.
Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7 x 8 equals the well-remembered 7 x 5 + 7 x 3, in preparation for learning about the distributive property. In the expression $\mathbf{x}^2 + 9\mathbf{x} + 14$, older students can see the 14 as 2 x 7 and the 9 as 2 + 7. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(\mathbf{x} - \mathbf{y})^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers **x** and **y**.

MP.8 Look for and express regularity in repeated reasoning.
Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation (y – 2)/(x – 1) = 3. Noticing the regularity in the way terms cancel when expanding (x – 1)(x + 1), (x – 1)(x2 + x + 1), and (x – 1)(x3 + x2 + x + 1) might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

Problem of the Month

Perfect Pair

Task Description – Primary Level

This task challenges students to name and define attributes of two-dimensional shapes [triangles, squares, rhombi, trapezoids, and/or hexagons]. This task also challenges students to visualize a figure with two different orientations from a set of shapes [triangles, squares, rhombi, trapezoids, and/or hexagons]. This development of spatial sense supports the developing understanding of attributes in plane geometry and the use of basic transformations.

Common Core State Standards Math - Content Standards

Operations and Algebraic Thinking

Understand addition as putting together and adding to, and understand subtraction as taking apart and taking from.

K.OA.1. Represent addition and subtraction with objects, fingers, mental images, drawings, sounds (e.g., claps), acting out situations, verbal explanations, expressions, or equations.

K.OA.2. Solve addition and subtraction word problems, and add and subtract within 10, e.g., by using objects or drawings to represent the problem.

K.OA.3. Decompose numbers less than or equal to 10 into pairs in more than one way, e.g., by using objects or drawings, and record each decomposition by a drawing or equation (e.g., 5 = 2 + 3 and 5 = 4 + 1).

Common Core State Standards Math – Standards of Mathematical Practice

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